The BKL proposal and cosmic censorship

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References: Gowdy phenomenology in scale-invariant variables, CQG 21 (2004) S29-S57; gr-qc/0310127

Asymptotic silence of generic cosmological singularities, Phys. Rev. Lett. 94 (2005) 051101; gr-qc/0402051

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Spacetimes, singularities, censorship

- Consider spacetimes $(V, g_{\alpha\beta})$, signature $- + + \cdots +$.
- $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}$
- assume $(V, g_{\alpha\beta})$ maximal, globally hyperbolic, energy conditions
- *Singularity theorems* $\Rightarrow$ generic spacetimes are causally geodesically incomplete (singular), but give no information about the nature of the singularities.
- The strong *Cosmic Censorship Conjecture* states that generic maximal globally hyperbolic spacetimes are inextendible.

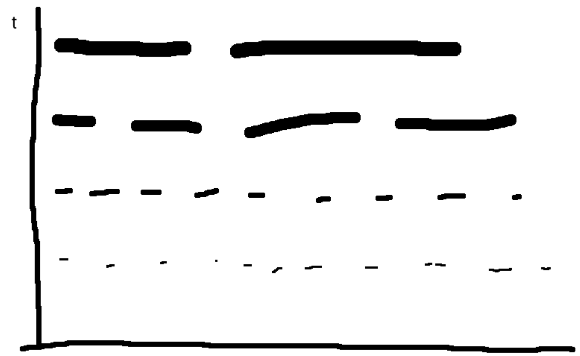
BKL proposal

- *Belinskiĭ, Khalatnikov and Lifshitz (BKL) proposal:* heuristic scenario for generic cosmological singularities
- The singularity is *spacelike*: observers near the singularity can't have communicated in the past; *silence* holds — particle horizons shrink to zero.
- The singularity is *local*: spatial derivatives are dynamically insignificant near the singularity

BKL proposal - cont.

- non-stiff matter is dynamically insignificant near the singularity
- The singularity is oscillatory in case matter is non-stiff and $D < 11$ and non-oscillatory otherwise.
- non-oscillatory — AVTD — asymptotically Kasner along generic timelines
- oscillatory — Kasner epochs interspersed with bounces which change the Kasner parameters according to BKL map.

Silent, oscillatory



Dynamical systems approach

Use scale invariant (Hubble normalized) frame variables:

- first order form of evolution equations
- classify fixed point sets, attractors etc.
- natural formulation of BKL proposal
- asymptotic dynamical system (silent boundary system)

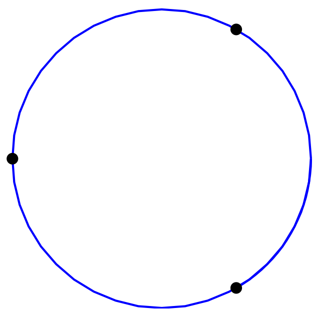
- Consider G_2 case on T^3 with nonzero twist.

- Use RNPL to study evolution numerically.

- The silent boundary system governs the evolution for generic timelines.

Example: Kasner

- $ds^2 = -dt^2 + t^{2p}dx^2 + t^{2q}dy^2 + t^{2r}dz^2$
- Vacuum ($R_{\alpha\beta} = 0$) implies the *Kasner relations*: $p + q + r = 1$, $p^2 + q^2 + r^2 = 1 \Rightarrow$ sphere intersected by plane $\Rightarrow$ unit circle in Σ_+, Σ_- plane, $\Sigma_+ = \frac{3}{2}(q + r) - 1$, $\Sigma_- = \frac{\sqrt{3}}{2}(q - r)$.
- Permutations of $p, q, r \Leftrightarrow 2\pi/3$ rotations of Σ-plane.
- Flat Kasner solutions correspond to $(p, q, r) = (1, 0, 0)$ and permutations thereof $\Leftrightarrow$ special points T_1, T_2, T_3 in the Kasner circle, $(-1, 0), (\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.



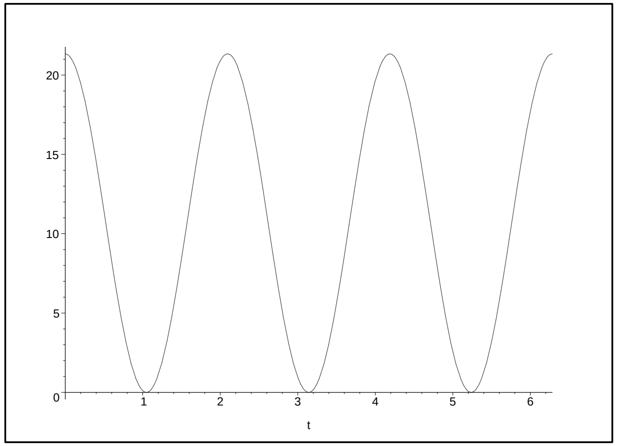
The Kasner circle $\mathcal{K} = \{\Sigma_+^2 + \Sigma_-^2 = 1\}$

Kasner – cont.

Rescaled Weyl tensor components

$$\mathcal{E}_+ = \frac{1}{3}\left((1+\Sigma_+)\,\Sigma_+ - \Sigma_-^2\right), \qquad \mathcal{E}_- = \frac{1}{3}\,(1-2\Sigma_+)\,\Sigma_-$$

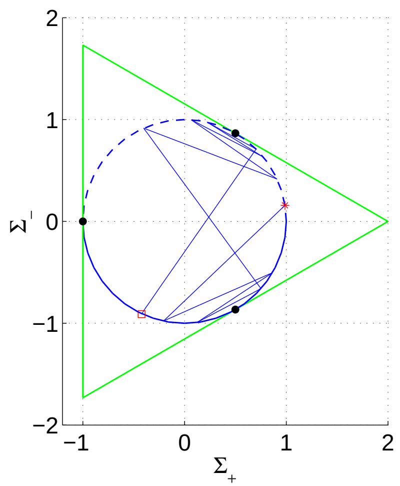
Weyl scalar $\mathcal{I}_1 = 48(\mathcal{E}_+^2 + \mathcal{E}_-^2)$

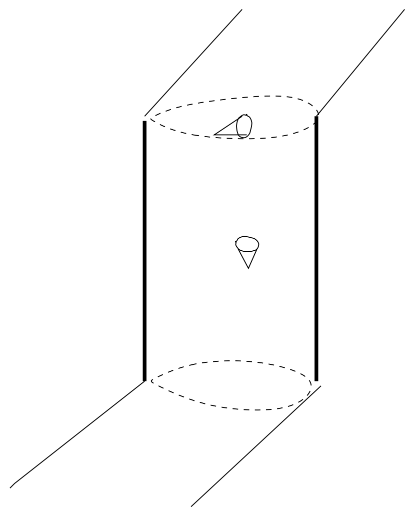


Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \sim t^{-4}\mathcal{I}_1$ blows up as $t \searrow 0 \Rightarrow$ nonflat Kasners are inextendible.

Example: Bianchi

- spatially homogenous models $\Rightarrow$ Einstein equations become ODE's.
- Classify according to isometry group
- "generic" Bianchi models have oscillatory singularity



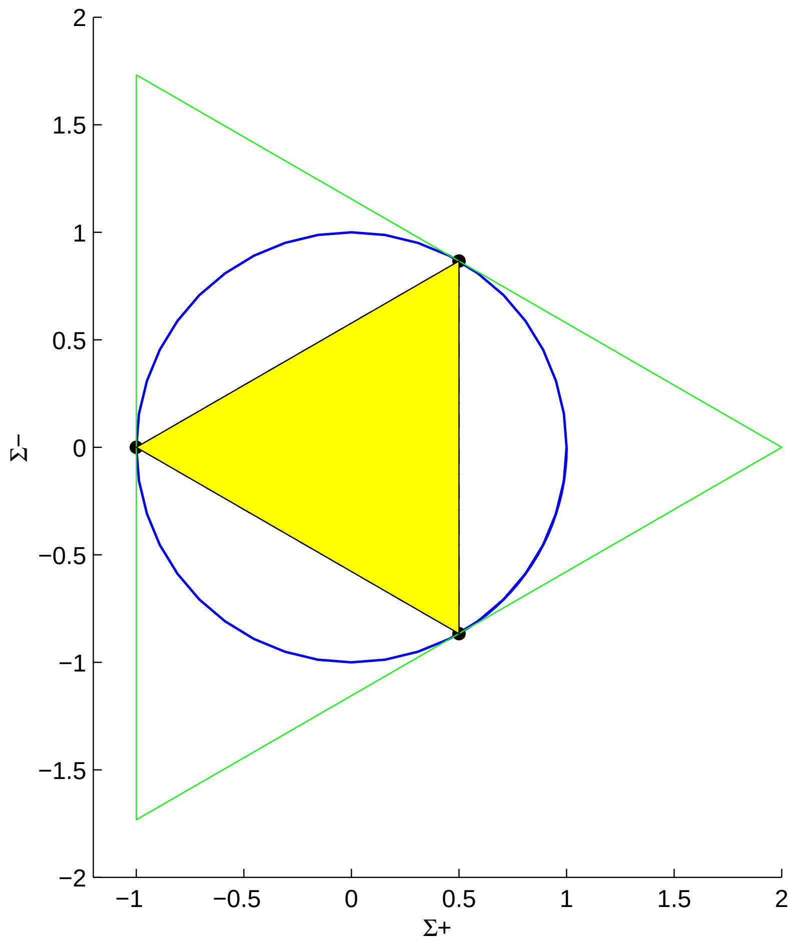


(a) (b)

(a) Kasner billiard — Bianchi IX — Mixmaster

(b) Taub-NUT has Cauchy horizon — extendible.

Example: Bianchi with stiff fluid



Stable region for Bianchi IX with stiff fluid.

BKL map

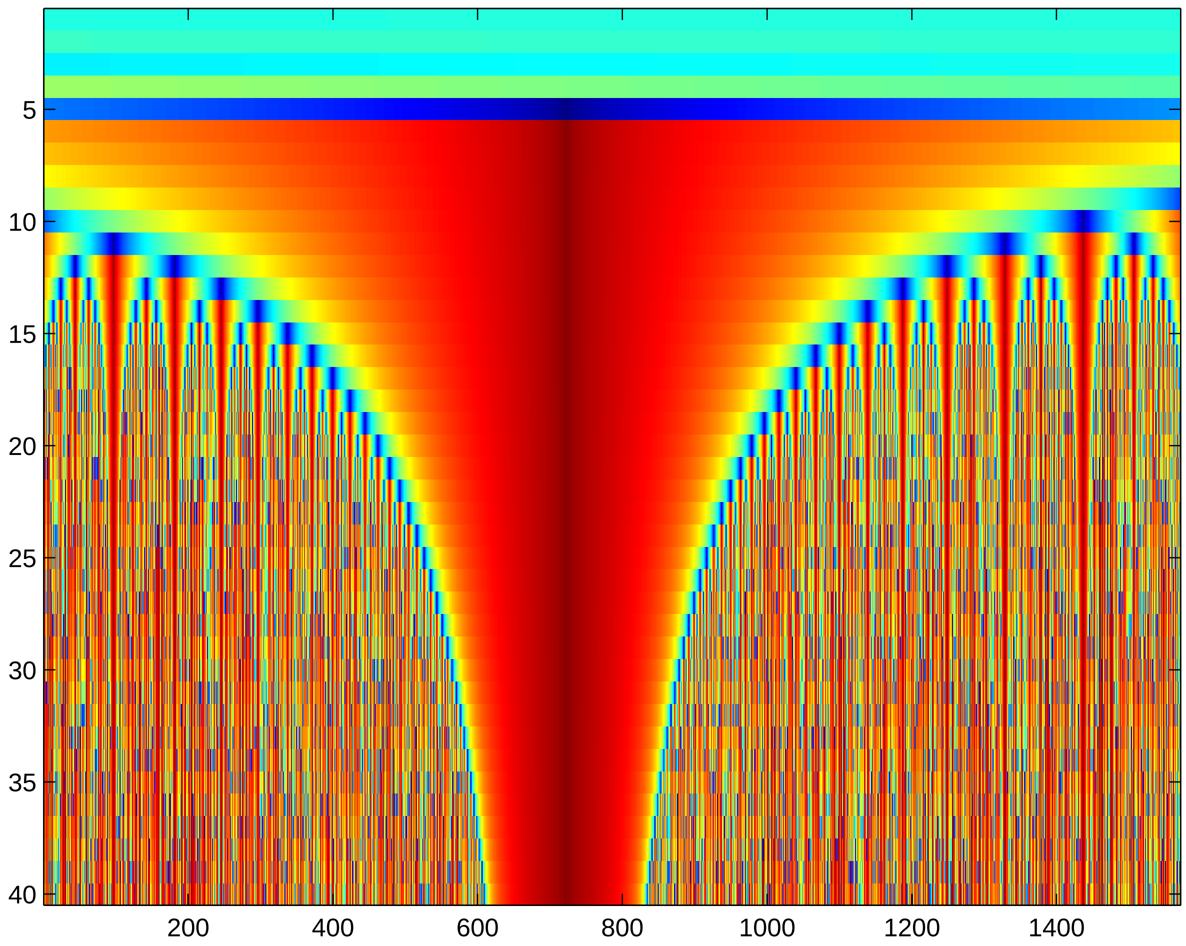
$$p = \frac{1+u}{1+u+u^2}$$
$$q = \frac{-u}{1+u+u^2}$$
$$r = \frac{u+u^2}{1+u+u^2}$$

BKL observed that the (chaotic!) map

$$u \mapsto \begin{cases} u-1 & u>1 \\ 1/u & 0<u<1 \end{cases}$$

is a good model for the asymptotic dynamics of the Kasner exponents in the case of Bianchi IX.

BKL map image



Hierarchy of cosmological models

orbit dimension	**system**	**type**
3	Bianchi or K-S	ODE
2	Surface symmetry or G_2	1+1 PDE
1	G_1	2+1 PDE
0	G_0	3+1 PDE

Example: Gowdy

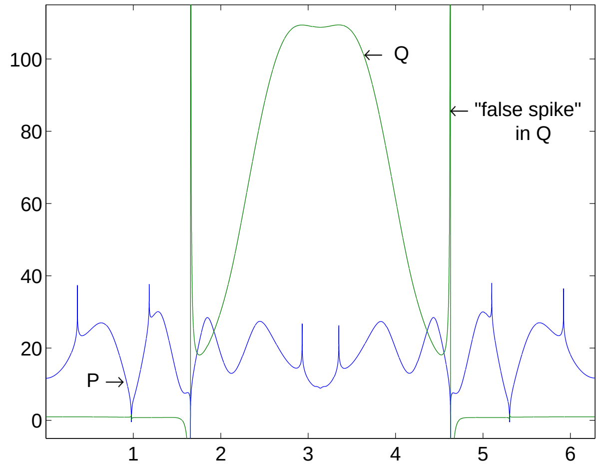
- Generic Gowdy spacetimes have AVTD singularity, cosmic censorship holds (Ringström, 2004).
- AVTD solutions for Gowdy

$$P(t,x) = k(x)t + \phi(x) + e^{-\epsilon t}u(t,x)$$

$$Q(t,x) = q(x) + e^{-2k(x)t}[\psi(x) + w(t,x)]$$

where $\epsilon > 0$, $u, w \to 0$ as $t \to \infty$ and $0 < k < 1$.

- spikes form in generic Gowdy (Berger and Garfinkle, 1998)



Chaos in superstring cosmology

(Damour and Henneaux, 2001; Damour et al., 2003)

- The singular (BKL) limit of D-dimensional gravity, including dilaton and form fields can be represented as a geodesic billiard in hyperbolic space, cf. Misner-Chitre model for Mixmaster.
- The billiard table is the Weyl chamber of a Lorentzian Kac-Moody algebra.
- Low-energy bosonic sector of superstring/M-theory models gives $D = 11$ dimensional gravity coupled to dilaton and p-form fields.
- The billiards corresponding to the $D = 10$ string theories (M, IIA, IIB, I, HO, HE) are of arithmetical type.

Dynamical systems approach: connection variables

Introduce a group-invariant orthonormal frame $\{\, \boldsymbol{e}_a \,\}_{a=0,1,2,3}$, with $\boldsymbol{e}_0$ timelike, align $\boldsymbol{e}_2$ with one of the Killing fields.

The nonzero connection variables for G_2 are:

- Θ, the volume expansion rate, $H := \frac{1}{3}\,\Theta$,
- σ_+, σ_-, $\sigma_\times$, σ_2; shear
- n_- and $n_\times$, commutation functions, giving the spatial connection on $\mathbb{T}^3$,
- $\dot{u}_1$, the acceleration of the integral curves of $\boldsymbol{e}_0$,
- q,r; deceleration parameter and logarithmic spatial Hubble gradient,
- $N^{-1}\partial_t$ and ${e_1}^1\partial_x$ are nontrivial derivatives on coordinate scalars.

Hubble normalized variables

$$(\mathcal{N}^{-1},\, E_1{}^1\,) := (\,N^{-1},\, e_1{}^1\,)/H$$

$$(\,\Sigma_{\ldots},\, N_{\ldots},\, \dot{U}) := (\,\sigma_{\ldots},\, n_{\ldots},\, \dot{u}_1)/H \;\; .$$

State vector:

$$\boldsymbol{X} = (E_1{}^1, \Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_2, N_\times, N_-)^T = (E_1{}^1) \otimes \boldsymbol{Y} \;\; .$$

To write evolution equations, need deceleration parameter q and logarithmic spatial Hubble gradient r,

$$(q+1) := -\,\mathcal{N}^{-1}\,\partial_t \ln(H) \;\; ,$$

$$r := -\,E_1{}^1\,\partial_x \ln(H) \;\; ,$$

where q and r satisfy the integrability condition

$$\mathcal{N}^{-1}\,\partial_t r - E_1{}^1\,\partial_x q = (q + 2\Sigma_+)\,r - (r - \dot{U})\,(q+1) \;\; .$$

Hubble normalized system of equations for G_2

Constraints:

$$
\begin{aligned}
(r - \dot{U}) &= {E_1}^1\, \partial_x \ln(1 + \Sigma_+) \\
1 &= \Sigma_+^2 + \Sigma_2^2 + \Sigma_-^2 + N_\times^2 + \Sigma_\times^2 + N_-^2 \\
(1 + \Sigma_+)\, \dot{U} &= -\,3(N_\times\, \Sigma_- - N_-\, \Sigma_\times) \\
0 &= ({E_1}^1\, \partial_x - r + \sqrt{3} N_\times)\, \Sigma_2 \ .
\end{aligned}
$$

Hubble normalized evolution equations for G_2

$$C^{-1}(1+\Sigma_+)\,\partial_t E_1{}^1 = (q+2\Sigma_+)\,E_1{}^1$$

$$C^{-1}(1+\Sigma_+)\,\partial_t(1+\Sigma_+) = (q-2)\,(1+\Sigma_+) + 3\Sigma_2^2$$

$$C^{-1}(1+\Sigma_+)\,\partial_t \Sigma_2 = (q-2-3\Sigma_+ + \sqrt{3}\Sigma_-)\,\Sigma_2$$

$$C^{-1}(1+\Sigma_+)\,\partial_t \Sigma_- + E_1{}^1\,\partial_x N_\times = (q-2)\,\Sigma_- + (r-\dot{U})\,N_\times + 2\sqrt{3}\Sigma_\times^2 - 2\sqrt{3}N_-^2 - \sqrt{3}\Sigma_2^2$$

$$C^{-1}(1+\Sigma_+)\,\partial_t N_\times + E_1{}^1\,\partial_x \Sigma_- = (q+2\Sigma_+)\,N_\times + (r-\dot{U})\,\Sigma_-$$

$$C^{-1}(1+\Sigma_+)\,\partial_t \Sigma_\times - E_1{}^1\,\partial_x N_- = (q-2-2\sqrt{3}\Sigma_-)\,\Sigma_\times - (r-\dot{U}+2\sqrt{3}N_\times)\,N_-$$

$$C^{-1}(1+\Sigma_+)\,\partial_t N_- - E_1{}^1\,\partial_x \Sigma_\times = (q+2\Sigma_+ + 2\sqrt{3}\Sigma_-)\,N_- - (r-\dot{U}-2\sqrt{3}N_\times)\,\Sigma_\times \ ,$$

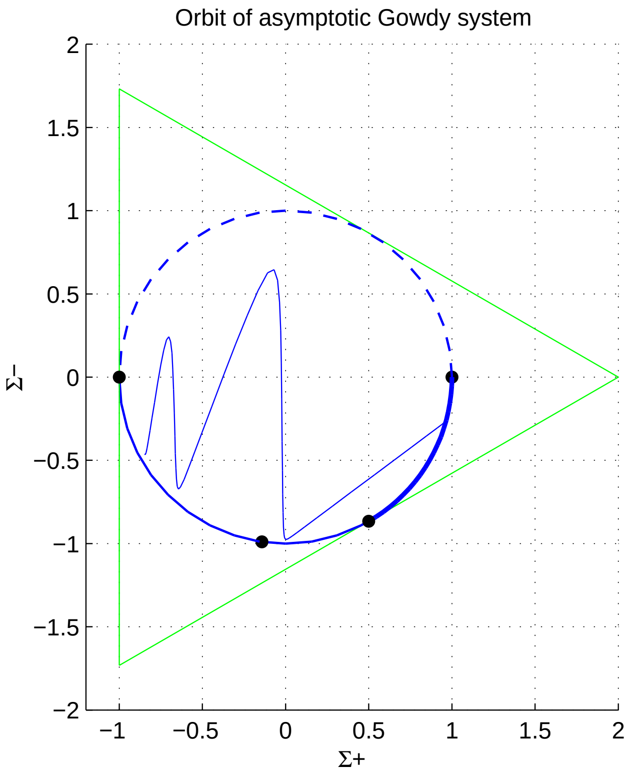
where $\quad q := 2(\Sigma_+^2 + \Sigma_-^2 + \Sigma_\times^2 + \Sigma_2^2) - \frac{1}{3}\,(E_1{}^1\,\partial_x - r + \dot{U})\,\dot{U}\ .$

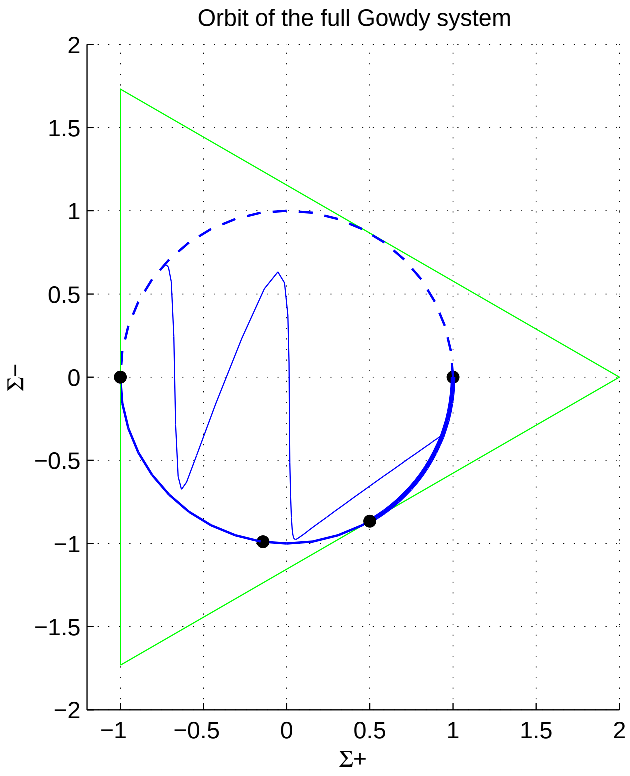
The silent boundary

- Observe numerically that ${E_1}^1 \to 0$ exponentially as $t \to \infty$.
- The unphysical boundary of the phase space with ${E_1}^1 = 0$ is called the *silent boundary*. The spatial derivative in the Hubble normalized system is of the form ${E_1}^1 \partial_x$. Therefore going to the silent boundary corresponds to collapse of the light cones.
- The dynamics on the silent boundary gives an *asymptotic dynamical system*, the SB system.
- The SB system is equivalent to a *spatially self-similar* model.

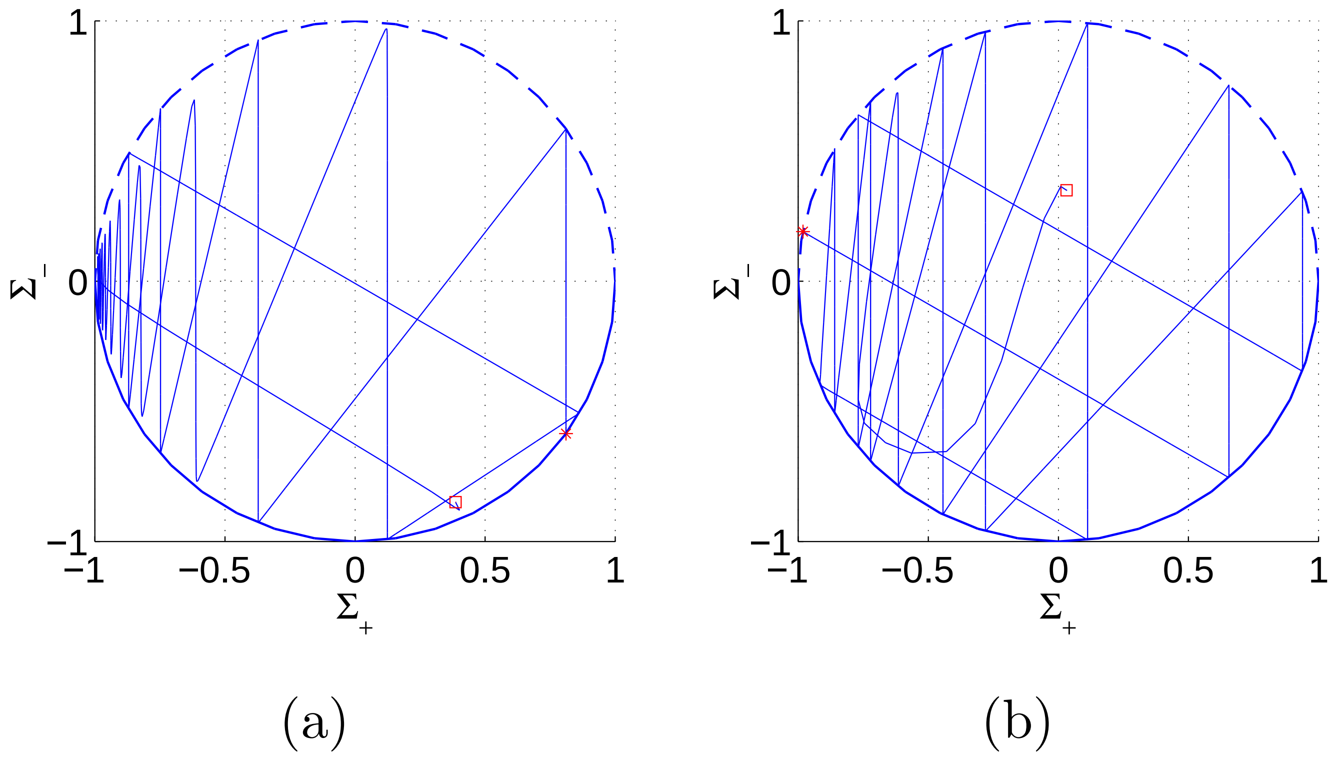
Gowdy – non-oscillatory singularity

- Setting $\Sigma_2 = 0$ gives the Gowdy subcase.
- The solution approaches stable arc, except for the spike timelines.



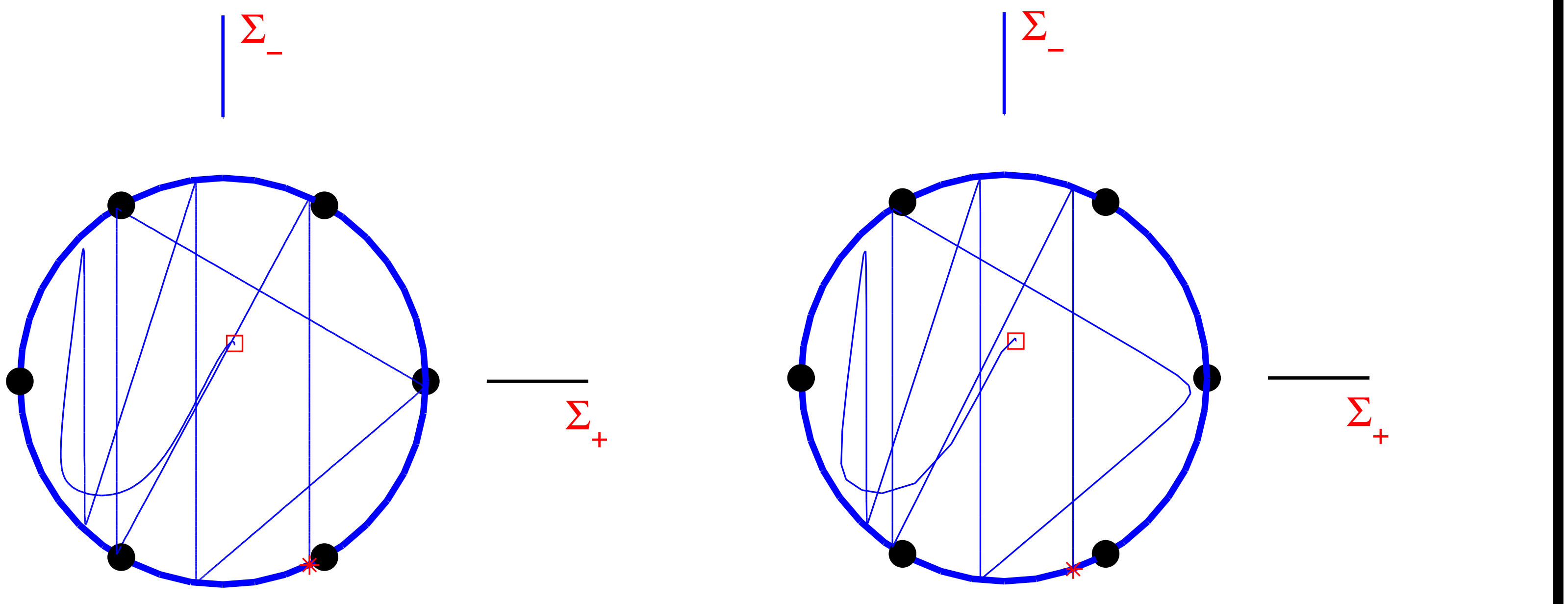


Bianchi $\mathrm{VI}^*_{-1/9}$ and the G_2 silent boundary



(a) The exceptional Bianchi type $\mathrm{VI}^*_{-1/9}$ system
(b) The G_2 silent boundary system: exceptional SSS Type-${}_{-1}\mathrm{VI}_0$

G_2 and silent boundary system



(a) (b)

Projections onto the $(\Sigma_+\Sigma_-)$-plane of a typical timeline for (a) the full G_2 system, and (b) its restriction to the silent boundary.

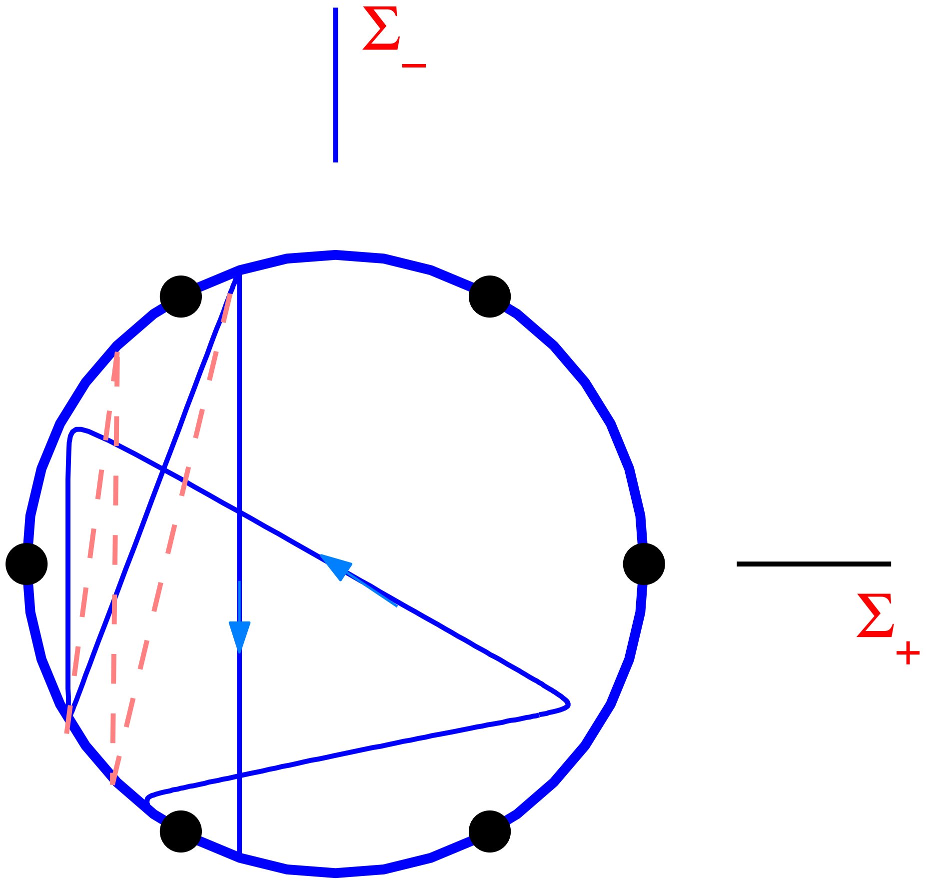
Spike timelines

- $N_-, \Sigma_2, \Sigma_\times$ are unstable on $\mathcal{K}$
- zero crossings of unstable variables ↔ "spiky features"
- Σ_2 cannot cross zero for generic solution
- $\Sigma_\times$ spikes "false" (gauge) spikes
- N_- spikes "true" (physical) spikes

$$E_1{}^1 \partial_x N_- \propto \hat{E}_1{}^1 \left[\partial_x \hat{N}_- + t\, \hat{N}_-\, \partial_x k(x)\right] e^{-[2-k(x)]t} \ .$$

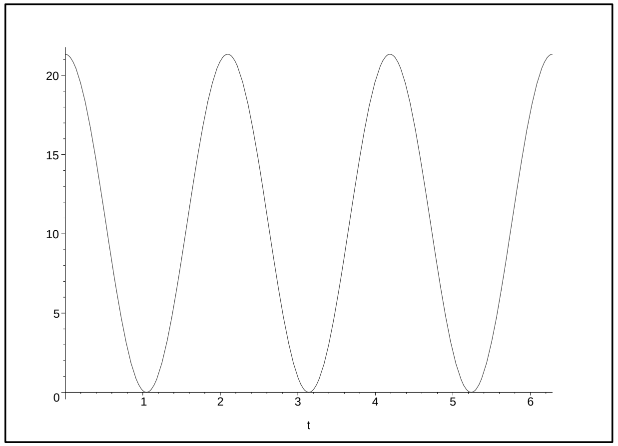
where $k(x) = -\sqrt{3}\hat{\Sigma}_-(x)/[1+\hat{\Sigma}_+(x)]$.

A spike timeline



Weyl tensor

- Timelines (x, t) are non-spike or spike.
- For non-spike timelines ${E_1}^1 \partial_x \boldsymbol{Y} \to 0$
- Each timeline revisits the Kasner circle $\mathcal{K}$, where $\Sigma_+^2 + \Sigma_-^2 = 1$, $N_\times = N_- = \Sigma_\times = \Sigma_2 = 0$
- On $\mathcal{K}$ only the rescaled Weyl tensor components $\mathcal{E}_+$ and $\mathcal{E}_-$ are non-vanishing, so the rescaled Kretschmann scalar is $\mathcal{I} = 48(\mathcal{E}_+^2 + \mathcal{E}_-^2)$.



Weyl tensor cont.

- For spike timelines, get contributions to Weyl tensor from ${E_1}^1\partial_x N_-$ and ${E_1}^1\partial_x \Sigma_\times$ in $\mathcal{E}_\times$ and $\mathcal{H}_-$ respectively.
- Thus, in addition to $\mathcal{E}_+$, $\mathcal{E}_-$ we have $\mathcal{E}_\times$ and $\mathcal{H}_-$ active at $\mathcal{K}$ (but not simultaneously!).
- Therefore expect the rescaled Kretschmann scalar $\mathcal{I} = 8(\mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta} - \mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta})$ be be nonzero for a sequence of times even along spike timelines.
- This supports cosmic censorship for generic $G2$ spacetimes.

Concluding remarks

- The dynamical systems approach using Hubble normalized variables gives a natural asymptotic dynamical system, the *silent boundary system*, and allows the analysis of the asymptotic dynamics in terms of *attractors*.
- In Bianchi IX, the Bianchi II attractor explains the BKL map (Kasner billiard) oscillatory asymptotic behavior.
- Gowdy has a stable attractor in $\mathcal{K}$ – so is AVTD and censorship holds.
- For the G_2 system, the silent boundary system explains the BKL map oscillatory asymptotic behavior.
- The G_2 silent boundary system (and therefore generic timelines of G_2) revisits the Kasner circle infinitely often — censorship holds.

Concluding remarks - cont.

- Next candidate for rigorous proof of censorship: G2KG = G_2 with scalar field.
- Construct spike solutions for G2KG.
- Asymptotic expansions near the silent boundary?
- Verify "asymptotic silence" numerically for U(1) and G_0 models?
- Role of "spikes" in stringy gravity?

References

Andersson, L. and Rendall, A. D. (2001). Quiescent cosmological singularities. *Comm. Math. Phys.*, 218(3):479–511.

Berger, B. K. and Garfinkle, D. (1998). Phenomenology of the Gowdy universe on $T^3 \times R$. *Phys. Rev. D*, 57:4767–4777.

Damour, T. and Henneaux, M. (2001). $E_{(10)}$, $BE_{(10)}$ and arithmetical chaos in superstring cosmology. *Phys.Rev.Lett.*, 86:4749–4752.

Damour, T., Henneaux, M., and Nicolai, H. (2003). Cosmological billiards. *Classical Quantum Gravity*, 20(9):R145–R200.

Ringström, H. (2000). Curvature blow up in Bianchi VIII and IX vacuum spacetimes. *Classical Quantum Gravity*, 17(4):713–731.

Ringström, H. (2004). Strong cosmic censorship in T^3-Gowdy spacetimes.