FINITE ELEMENT METHODS FOR HYPERBOLIC SYSTEMS

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Outline of Talk

1. Methods for a simple scalar problem

2. Hyperbolic Systems

3. Numerical methods for problems with no specific time dependence

4. Numerical methods for problems with explicit time dependence

Scalar Problem

Let Ω – bounded polygonal domain in $\mathbb{R}^2$.

Consider simple linear hyperbolic problem:

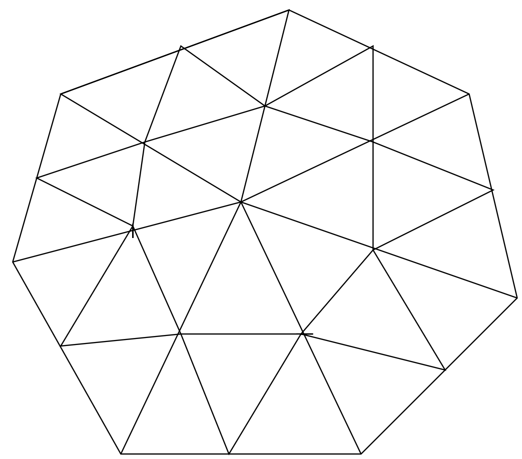
$$\boxed{\boldsymbol{\alpha} \cdot \nabla u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma_{in}(\Omega),}$$

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ constant vector,
$\Gamma_{in}(\Omega)$ portion of $\partial\Omega$ on which $\boldsymbol{\alpha} \cdot \boldsymbol{n} < 0$,
$\boldsymbol{n} =$ unit outward normal to $\partial\Omega$.

Fundamental idea of finite element methods:

Place mesh of triangles (or other elements) on Ω
(assume max diameter of elements $\leq h$).

Triangular Mesh on polygon Ω.



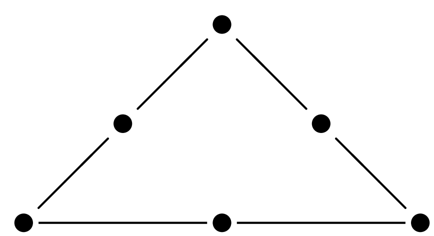
Seek approximate solution u_h in space of piecewise polynomials.

Determine u_h by appropriate variational formulation of problem.

Many ways in which this can be done.

Consider triangle T with one side on inflow boundary $\Gamma_{in}(\Omega)$. Assume $u_h \in P_2(T)$.

Then u_h has 6 degrees of freedom, which may be taken to be values at vertices and values at midpoints of edges (or average value of u_h on edges).



Continuous Galerkin method: Reed-Hill 1973
(strongly imposed boundary conditions):

Let $u_h = g_I$ on $\Gamma_{in}(T)$, g_I a quadratic interpolant of g. Leaves 3 degrees of freedom to be determined.

Observe true solution u satisfies:

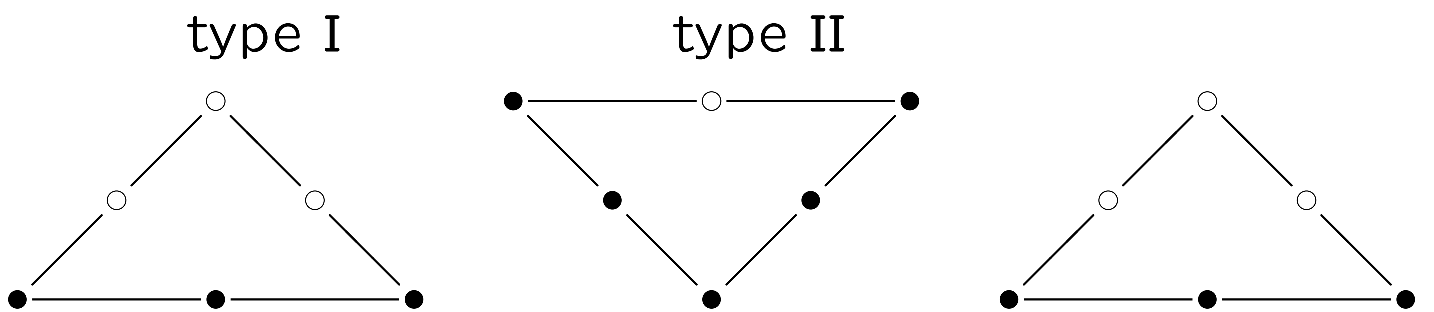
$$\int_T \boldsymbol{\alpha} \cdot \nabla u \, v \, dx = \int_T f \, v \, dx$$

So could determine u_h by equation:

$$\int_T \boldsymbol{\alpha} \cdot \nabla u_h v \, dx = \int_T f v \, dx, \qquad \forall v \in P_1(T).$$

Use this equation to determine approximate solution in all triangles with one-inflow side on inflow boundary (can be done in parallel on all such triangles).

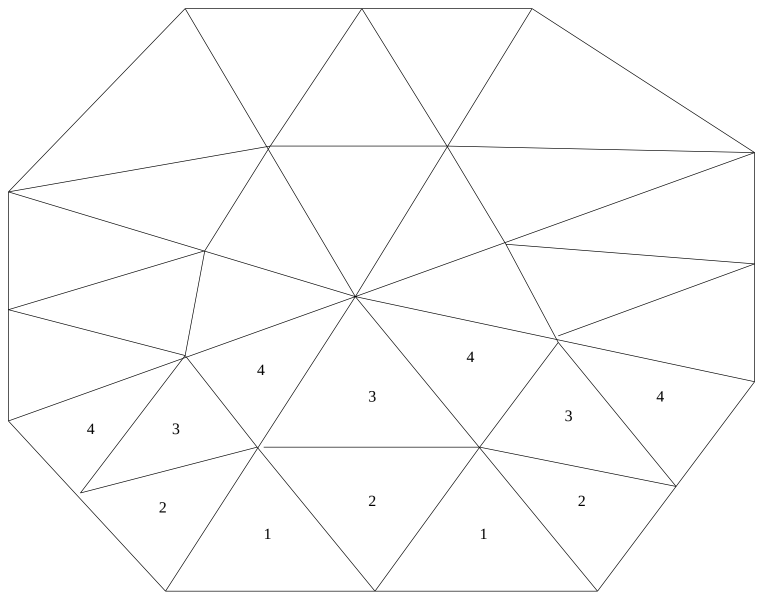
Then must solve on triangles with 2 inflow sides:



By continuity, u_h already known at 5 degrees of freedom on type II triangle. Determine remaining degree of freedom by:

$$\int_T \boldsymbol{\alpha} \cdot \nabla u_h v \, dx = \int_T f v \, dx, \qquad \forall v \in P_0(T).$$

Computation of Approximate Solution



Discontinuous Galerkin method: Reed-Hill 1973
(weakly imposed boundary conditions):

Integrating by parts:

$$\int_T \boldsymbol{\alpha} \cdot \nabla u v \, dx = \oint_{\partial T} u v \boldsymbol{\alpha} \cdot n - \int_T u \boldsymbol{\alpha} \cdot \nabla v \, dx$$
$$= \int_{\Gamma_{out}(T)} u v \boldsymbol{\alpha} \cdot n + \int_{\Gamma_{in}(T)} g v \boldsymbol{\alpha} \cdot n - \int_T u \boldsymbol{\alpha} \cdot \nabla v \, dx$$

So can now determine u_h by equation:

$$\int_{\Gamma_{out}(T)} u_h v \boldsymbol{\alpha} \cdot n - \int_T u_h \boldsymbol{\alpha} \cdot \nabla v \, dx$$
$$= \int_T f v \, dx - \int_{\Gamma_{in}(T)} g v \, \boldsymbol{\alpha} \cdot n, \quad \forall v \in P_2(T).$$

Can rewrite this in form:

$$\oint_{\partial T} u_h v \, \boldsymbol{\alpha} \cdot n - \int_T u_h \, \boldsymbol{\alpha} \cdot \nabla v \, dx$$
$$= \int_T f v \, dx + \int_{\Gamma_{in}(T)} (u_h - g) v \, \boldsymbol{\alpha} \cdot n, \quad \forall v \in P_2(T),$$

or integrating back by parts:

$$\int_T \boldsymbol{\alpha} \cdot \nabla u_h \, v \, dx = \int_T f v \, dx + \int_{\Gamma_{in}(T)} (u_h - g) v \boldsymbol{\alpha} \cdot n$$

To continue inside Ω, use similar approach, but write equation as:

$$\int_T \boldsymbol{\alpha} \cdot \nabla u_h v \, dx = \int_T f v \, dx + \int_{\Gamma_{in}(T)} (u_h^+ - u_h^-) v \, \boldsymbol{\alpha} \cdot n$$

for all $v \in P_2(T)$, where

$$v^{\pm}(x) = \lim_{\epsilon \to 0^{\pm}} v(x + \epsilon \boldsymbol{\alpha}),$$

i.e., u_h^+ denotes u_h inside T and u_h^- denotes value of u_h inside T^- (already known).

Basic conservation property of homog. equation.

Multiplying by u and integrating over subdomain G,

$$0 = (\boldsymbol{\alpha} \cdot \nabla u, u)_G = \frac{1}{2} \int_G \boldsymbol{\alpha} \cdot \nabla (u^2) = \frac{1}{2} \int_{\partial G} u^2 \, \boldsymbol{\alpha} \cdot \boldsymbol{n}.$$

This may be written in the form

$$\boxed{\frac{1}{2}\int_{\Gamma_{out}(G)} u^2|\boldsymbol{\alpha}\cdot\boldsymbol{n}| = \frac{1}{2}\int_{\Gamma_{in}(G)} u^2|\boldsymbol{\alpha}\cdot\boldsymbol{n}|}$$

since $\boldsymbol{\alpha}\cdot\boldsymbol{n} \geq 0$ on $\Gamma_{out}(G)$ and $\boldsymbol{\alpha}\cdot\boldsymbol{n} \leq 0$ on $\Gamma_{in}(G)$.

If Ω disjoint union of subdomains G_i, summing identities, and cancellation of integrals over common boundaries leads to conservation result:

$$\boxed{\frac{1}{2}\int_{\Gamma_{out}(\Omega)} u^2|\boldsymbol{\alpha}\cdot\boldsymbol{n}| = \frac{1}{2}\int_{\Gamma_{in}(\Omega)} u^2|\boldsymbol{\alpha}\cdot\boldsymbol{n}|.}$$

Follow this type of analysis at discrete level to obtain stability and error analysis of finite element approximation schemes.

The Classical Discontinuous Galerkin Method

Let τ_h denote triangulation of Ω into triangles T, diameter $\leq h$, $P_n(T)$ space of polynomials of degree $\leq n$ on T.

For each $T \in \tau_h$, Discontinuous Galerkin method is:
Find $u_h \in P_n(T)$ such that

$$(\boldsymbol{\alpha} \cdot \nabla u_h, v_h)_T - \int_{\Gamma_{in}(T)} (u_h^+ - u_h^-)\, v_h\, \boldsymbol{\alpha} \cdot \boldsymbol{n} = (f, v_h)_T$$

for all $v_h \in P_n(T)$, where $(\cdot, \cdot)_T$ denotes L^2 inner product over T.

Following continuous problem, take test function $v_h = u_h$. Then, for homogeneous problem, $f = 0$

$$(\boldsymbol{\alpha} \cdot \nabla u_h, u_h)_T - \int_{\Gamma_{in}(T)} (u_h^+ - u_h^-)\, u_h^+\, \boldsymbol{\alpha} \cdot \boldsymbol{n} = 0.$$

Integrating first term to boundary, get

$$\frac{1}{2}\oint_{\partial T} u_h^2 \boldsymbol{\alpha}\cdot\boldsymbol{n} - \int_{\Gamma_{in}(T)} (u_h^+ - u_h^-)\, u_h^+\, \boldsymbol{\alpha}\cdot\boldsymbol{n}$$

$$= \frac{1}{2}\int_{\Gamma_{out}(T)} (u_h^-)^2 \boldsymbol{\alpha}\cdot\boldsymbol{n} + \frac{1}{2}\int_{\Gamma_{in}(T)} (u_h^+)^2 \boldsymbol{\alpha}\cdot\boldsymbol{n}$$

$$- \int_{\Gamma_{in}(T)} (u_h^+ - u_h^-)\, u_h^+\, \boldsymbol{\alpha}\cdot\boldsymbol{n}$$

$$= \frac{1}{2}\oint_{\partial T} (u_h^-)^2 \boldsymbol{\alpha}\cdot\boldsymbol{n} - \frac{1}{2}\int_{\Gamma_{in}(T)} (u_h^-)^2 - (u_h^+)^2] \boldsymbol{\alpha}\cdot\boldsymbol{n}$$

$$- \int_{\Gamma_{in}(T)} [(u_h^+)^2 - u_h^- u_h^+] \boldsymbol{\alpha}\cdot\boldsymbol{n}$$

$$= \frac{1}{2}\oint_{\partial T} (u_h^-)^2 \boldsymbol{\alpha}\cdot\boldsymbol{n} - \frac{1}{2}\int_{\Gamma_{in}(T)} [u_h^+ - u_h^-]^2 \boldsymbol{\alpha}\cdot\boldsymbol{n}$$

Local stability estimate:

$$\frac{1}{2}\int_{\Gamma_{out}(T)} (u_h^-)^2 |\boldsymbol{\alpha}\cdot\boldsymbol{n}| + \frac{1}{2}\int_{\Gamma_{in}(T)} [u_h^+ - u_h^-]^2 |\boldsymbol{\alpha}\cdot\boldsymbol{n}|$$
$$= \frac{1}{2}\int_{\Gamma_{in}(T)} (u_h^-)^2 |\boldsymbol{\alpha}\cdot\boldsymbol{n}|$$

Summing over triangles, $G = \cup T$,

$$\frac{1}{2}\int_{\Gamma_{out}(G)} (u_h^-)^2 |\boldsymbol{\alpha}\cdot\boldsymbol{n}| \sum_T \frac{1}{2}\int_{\Gamma_{in}(T)} (u_h^+ - u_h^-)^2 |\boldsymbol{\alpha}\cdot\boldsymbol{n}|$$
$$= \frac{1}{2}\int_{\Gamma_{in}(G)} (u_h^-)^2 |\boldsymbol{\alpha}\cdot\boldsymbol{n}|.$$

Basic identity to establish stability.

Remark: Get other DG methods by making other choices of u_h on $\Gamma_{in}(T)$ (than u_h^-).

Hyperbolic Systems

Let Ω_x – bounded domain in $\mathbb{R}^N$ with boundary Γ_x.

First order hyperbolic systems, positive in sense of Friedrichs

$$L\boldsymbol{u} \equiv A_0 \partial \boldsymbol{u}/\partial t + \sum_{i=1}^{N} A_i \partial \boldsymbol{u}/\partial x_i + B\boldsymbol{u} = \boldsymbol{F} \quad \text{in } \Omega,$$

$$(M - D)\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

where A_0 is either 0 or I, A_i, B, M given $m \times m$ matrices depending on $\boldsymbol{x}$, $\boldsymbol{u}$ an m-vector.

$\Omega = \Omega_x$ when $A_0 = 0$ and $\Omega = \Omega_x \times (0, T)$ when $A_0 = I$, $\Gamma =$ boundary of Ω.

$D = \sum_{i=0}^{N} A_i n_i$, where $\boldsymbol{n} = (n_i)$ is unit outward normal to Γ.

For mathematical analysis, let $\sigma > 0$ constant and assume:

$A_i(\boldsymbol{x})$ symmetric, $M + M^T \geq 0$ on Γ, $+$ other conditions

Then for $g = 0$

$$\int_\Omega \boldsymbol{F} \cdot \boldsymbol{u}\, d\boldsymbol{x} = \int_\Omega L\boldsymbol{u} \cdot \boldsymbol{u}\, d\boldsymbol{x} \geq \frac{\sigma}{2} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{u}\, d\boldsymbol{x} + \frac{1}{4} \int_\Gamma (M + M^T)\boldsymbol{u} \cdot \boldsymbol{u}\, ds \geq \frac{\sigma}{2} \|\boldsymbol{u}\|^2_{L^2(\Omega)}.$$

Examples:

Transport Equation: $m = 1$, $A_0 = 0$:

$$\boldsymbol{\beta} \cdot \nabla u + \gamma u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \Gamma_- = \{x \in \Gamma : \boldsymbol{\beta} \cdot \boldsymbol{n} < 0\}.$$

$D = \boldsymbol{\beta} \cdot \boldsymbol{n}$, $M = |D|$. Note $M - D = -2\boldsymbol{\beta} \cdot \boldsymbol{n}$ on Γ_- and $M - D = 0$ on Γ_+, so no boundary condition on outflow boundary.

Wave equation as first order system

Maxwell's equations

Numerical Methods: Case $A_0 = 0$, $\boldsymbol{g} = 0$

Let τ_h be family of "triangulations" of Ω_x into simplices indexed by h, maximum diameter of simplices $K \in \tau_h$.

$P_k(K)$ set of polynomials on K of total degree $\leq k$.

$V_h = \{v \in H^1(\Omega_x) : v|_K \in P_k(K) \quad \forall K \in \tau_h\}$.

$\boldsymbol{V}_h = [V_h]^m$.

$W_h = \{v \in L^2(\Omega_x) : v|_K \in P_k(K) \quad \forall K \in \tau_h\}$.

$\boldsymbol{W}_h = [W_h]^m$.

Set $(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_x} \boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x}$, $\qquad < \boldsymbol{u},\boldsymbol{v} >= \int_{\Gamma_x} \boldsymbol{u}\cdot\boldsymbol{v}\,ds$.

Standard Galerkin method: weakly imposed B.C.

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that $\forall \boldsymbol{v} \in \boldsymbol{V}_h$

$$(L\boldsymbol{u}_h,\boldsymbol{v}) + \frac{1}{2} < (M-D)\boldsymbol{u}_h,\boldsymbol{v} >= (\boldsymbol{F},\boldsymbol{v}).$$

Stability Estimate:

Exactly as in continuous case.

Choosing $\boldsymbol{v}_h = \boldsymbol{u}_h$,

$$\int_\Omega \boldsymbol{F} \cdot \boldsymbol{u}_h \, d\boldsymbol{x} = \int_\Omega L\boldsymbol{u}_h \cdot \boldsymbol{u}_h \, d\boldsymbol{x} + \frac{1}{2} < (M - D)\boldsymbol{u}_h, \boldsymbol{u}_h >$$
$$\geq \frac{\sigma}{2} \int_\Omega \boldsymbol{u}_h \cdot \boldsymbol{u}_h \, d\boldsymbol{x} + \frac{1}{4} \int_\Gamma (M + M^T)\boldsymbol{u}_h \cdot \boldsymbol{u}_h \, ds \geq \frac{\sigma}{2} \|\boldsymbol{u}_h\|^2_{L^2(\Omega)}.$$

Then

$$\|\boldsymbol{u}_h\|_{L^2(\Omega)} \leq C \|\boldsymbol{F}\|_{L^2(\Omega)}.$$

Error Estimate:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} \leq C h^k \|\boldsymbol{u}\|_{k+1}.$$

Streamline Diffusion Method:

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$:

$$(L\boldsymbol{u}_h, \boldsymbol{v} + hL_0\boldsymbol{v}) + \frac{1}{2} < (M - D)\boldsymbol{u}_h, \boldsymbol{v} >$$
$$= (\boldsymbol{F}, \boldsymbol{v} + hL_0\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h,$$

where $L_0 = \sum_{i=1}^{N} A_i \frac{\partial}{\partial x_i}$.

Improved Stability Estimate:

$$\sqrt{h}\|L_0\boldsymbol{u}_h\|_{L^2(\Omega)} + \|\boldsymbol{u}_h\|_{L^2(\Omega)} + < M\boldsymbol{u}_h, \boldsymbol{u}_h >^{1/2} \leq C\|\boldsymbol{F}\|_{L^2(\Omega)}$$

Improved Error Estimate:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} \leq Ch^{k+1/2}\|\boldsymbol{u}\|_{k+1}.$$

Discontinuous Galerkin Method:

Let $\delta = 0$ or h. Find $\boldsymbol{u}_h \in \boldsymbol{W}_h$:

$$\sum_{K \in \tau_h} [(L\boldsymbol{u}_h, \boldsymbol{v} + \delta L_0 \boldsymbol{v})_K + \frac{1}{2} < (M_K - D_K)[\boldsymbol{u}_h], \boldsymbol{v} >_K]$$

$$= \sum_{K \in \tau_h} (\boldsymbol{F}, \boldsymbol{v} + \delta L_0 \boldsymbol{v})_K, \quad \forall \boldsymbol{v} \in \boldsymbol{W}_h,$$

where $[\boldsymbol{u}] = \boldsymbol{u}^{int} - \boldsymbol{u}^{ext}$,

$$(\boldsymbol{u},\boldsymbol{v})_K = \int_K \boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x}, \quad <\boldsymbol{u},\boldsymbol{v}>_K = \int_{\partial K} \boldsymbol{u}\cdot\boldsymbol{v}\,ds,$$

$$\boldsymbol{u}^{int}(\boldsymbol{x}) = \lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\in K} \boldsymbol{u}(\boldsymbol{y}), \quad \boldsymbol{x}\in\partial K,$$

$$\boldsymbol{u}^{ext}(\boldsymbol{x}) = \lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\notin K} \boldsymbol{u}(\boldsymbol{y}), \quad \boldsymbol{x}\in\partial K,$$

$$D_K = \sum_{i=1}^{N} A_i n_i^K, \qquad M_K = \rho(D_K)I$$

Stability Estimate analogous to Streamline Diffusion, but norms are piecewise over the elements.

Error Estimate the same as for Streamline Diffusion.

In general, get global problem to solve.

Explicit Time Dependence: $A_0 = I$

$$L\boldsymbol{u} \equiv \partial \boldsymbol{u}/\partial t + \sum_{i=1}^{N} A_i \partial \boldsymbol{u}/\partial x_i + B\boldsymbol{u} = \boldsymbol{F}, \qquad \text{in } \Omega_x \times (0,T),$$

$$(M - D)\boldsymbol{u} = 0 \quad \text{on } \Gamma_x \times (0,T), \qquad \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_x,$$

where matrices A_i, B, M depend on $(\boldsymbol{x},t)$.

Since $\boldsymbol{n} = (\pm 1, 0, \ldots, 0)$ at $t = T$ and $t = 0$, respectively, $D = \sum_{i=0}^{N} A_i n_i = \pm I$.

Choosing $M = I$ for $t = 0$ or $t = T$,

$(M - D) = 0$ at $t = T$, so no BC there.

$(M - D) = 2I$ at $t = 0$, so get initial condition at $t = 0$.

Numerical Methods:

Apply Standard Galerkin or Streamline Diffusion method using finite element triangulations of strips $S_n = \Omega_x \times (t_n, t_{n+1})$.

Since $M - D = 0$ on "top" part of ∂S_n $(t = t_{n+1})$, compute discrete solution on one strip after another.

Let $V_h^n \subset H^1(S_n)$, $\boldsymbol{V}_h^n = [V_h^n]^m$, $\delta = 0$ or 1.

Find $\boldsymbol{u}_h^n \in \boldsymbol{V}_h^n$ such that $\forall \boldsymbol{v} \in \boldsymbol{V}_h^n$:

$$
\begin{aligned}
(L\boldsymbol{u}_h^n, \boldsymbol{v} + \delta h L_0 \boldsymbol{v})_{S_n} + \int_{\Omega_x} \boldsymbol{u}_+^n(\boldsymbol{x}, t_n) \cdot \boldsymbol{v}_+(\boldsymbol{x}, t_n)\, d\boldsymbol{x} \\
+ \frac{1}{2} < (M - D)\boldsymbol{u}_h, \boldsymbol{v} >_{\Gamma_x \times (t_n, t_{n+1})} \\
= (\boldsymbol{F}, \boldsymbol{v} + \delta h L_0 \boldsymbol{v})_{S_n} + \int_{\Omega_x} \boldsymbol{u}_-^n(\boldsymbol{x}, t_n) \cdot \boldsymbol{v}_+(\boldsymbol{x}, t_n)\, d\boldsymbol{x},
\end{aligned}
$$

where now $L_0 = \partial/\partial t + \sum_{i=1}^{N} A_i \partial/\partial x_i$.

Note: weakly imposed "continuity condition" at $t = t_n$:

$$
\begin{aligned}
\frac{1}{2}(M - D)\boldsymbol{u} \cdot \boldsymbol{v} &= \boldsymbol{u}_+^n(\boldsymbol{x}, t_n) \cdot \boldsymbol{v}_+(\boldsymbol{x}, t_n) \\
&= \boldsymbol{u}_-^n(\boldsymbol{x}, t_n) \cdot \boldsymbol{v}_+(\boldsymbol{x}, t_n).
\end{aligned}
$$

Another possibility: Discontinuous Galerkin on each strip.

Let $W_h^n \subset L^2(S_n)$, $\boldsymbol{W}_h^n = [W_h^n]^m$.

Use previous formulation, but replace

$$(L\boldsymbol{u}_h^n, \boldsymbol{v} + \delta L_0 \boldsymbol{v})_{S_n} \quad \text{by} \quad \sum_{K \in \tau_h \cap S_n} (L\boldsymbol{u}_h^n, \boldsymbol{v} + \delta L_0 \boldsymbol{v})_K$$

$$(\boldsymbol{F}, \boldsymbol{v} + \delta h L_0 \boldsymbol{v})_{S_n} \quad \text{by} \quad \sum_{K \in \tau_h \cap S_n} (\boldsymbol{F}, \boldsymbol{v} + \delta h L_0 \boldsymbol{v})_K$$

and add boundary integral terms

$$\sum_{K \in \tau_h \cap S_n} \frac{1}{2} < (M_K - D_K)[\boldsymbol{u}_h^n], \boldsymbol{v} >_K .$$

Note: Within each strip methods are implicit: must solve for all unknowns at the same time.

An Explicit Method

Divide space-time domain into mesh of elements K, union of simplices in $\mathbb{R}^{N+1}$.

Let $\Gamma^*(K) = \Gamma(K) \cap (\Gamma(\Omega_x) \times (0, T))$. On $\Gamma(K) - \Gamma^*(K)$, require $n_t \neq 0$ and define $D_t \equiv \text{sign}\{n_t\}\ D$.

Define bilinear form

$$a_K(\boldsymbol{u}, \boldsymbol{v}) \equiv (L\boldsymbol{u}, \boldsymbol{v})_K + \int_{\Gamma_{in}(K)} \Big[D_t \boldsymbol{u}\Big] \cdot \boldsymbol{v} + \frac{1}{2} \int_{\Gamma^*(K)} (M - D)\boldsymbol{u} \cdot \boldsymbol{v},$$

where, $\Gamma_{in}(K)$ [$\Gamma_{out}(K)$] denotes portion of $\Gamma(K) - \Gamma^*(K)$ where $n_t < 0$ [$n_t > 0$].

On each element K, approximate $\boldsymbol{u}$ by $\boldsymbol{u}_h \in \boldsymbol{P}_n(K)$, polynomials of total degree $\leq n$ over K.

Starting from appropriate interpolant $\boldsymbol{u}_h^-(x, 0)$ of initial condition $\boldsymbol{u}_0$, develop solution explicitly by:

$$a_K(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{F}, \boldsymbol{v})_K \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{P}_n(K).$$

Key idea: Special mesh construction allowing explicit method without violating domain of dependence considerations.