1. Well posed problems

a) First order systems, Cauchy problem

$$u_t = \sum_{j=1}^{s} A_j D_j u = P(D)u, \quad D_j = \partial/\partial x_j \qquad (1)$$

$$u(x,0) = f(x), \quad x = (x_1, \dots, x_s), \quad -\infty < x_j < \infty$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad A_j = \begin{pmatrix} a_{11}^{(j)} & \cdots & a_{1n}^{(j)} \\ \vdots & & \\ a_{n1}^{(j)} & \cdots & a_{nn}^{(j)} \end{pmatrix}.$$

Well posed

1) There is an estimate

$$\|u(\cdot, t)\| \le K e^{\alpha(t-t_0)} \|u(\cdot, t_0)\|$$

2) Perturbation by lower order terms
(If (1) is well posed so is $\tilde{u}_t = P(D)\tilde{u} + Bu$, for any bounded operator B)

$$\sum \varphi_j = 1 \qquad \varphi_j \in C_0^\infty$$

$$u^{(j)} = \varphi_j u$$

$$u_t^{(j)} = P(D) u^{(j)} + \text{lower order}$$

localised problems.

1) principle of frozen coefficients
2) Reduction to Halfplane problem and Cauchy problems.

<u>Theorem 1</u> $A_j = A_j^*$ the problem is well posed

If not let A_j be constant.

Simple wave solutions

$$u(x,t) = e^{i\langle\omega,x\rangle}\hat{u}(\omega,t), \quad \omega=(\omega_1,\dots,\omega_s) \text{ real}$$

Seperation of variables

$$\hat{u}_t = i|\omega|\,\hat{P}(\omega')\,\hat{u}, \quad \hat{P}(\omega') = \sum A_j\omega_j', \quad \omega' = \omega/|\omega| \qquad (2)$$

<u>Theorem 2</u> If the eigenvalues of $P(\omega')$ are not real then (2) has solutions which grow like

$$|\hat{u}(\omega,t)| \sim e^{|\omega|t}$$

If the eigenvalues are real (weakly hyperbolic) stability to lower order perturbation is not garantued.

The problem is well posed if and only if ~~the st~~ the system is strongly hyperbolic.

In this case the principle of frozen coefficients hold. (Smoothness assumptions for the symbol)

b) Second order systems.

$$u_{tt} = P_0 u + P_1 u_t$$

$$P_0 = \sum A_{ij} D_i D_j \qquad P_1 = \sum A_i D_i$$

Simple wave solutions

$$u(x,t) = e^{i\langle\omega,x\rangle}\hat{u}(\omega,t)$$

$$\hat{u}_{tt} = -|\omega|^2\hat{P}_0(\omega')\,\hat{u} + i|\omega|\,\hat{P}_1(\omega')\cdot\hat{u}_t$$

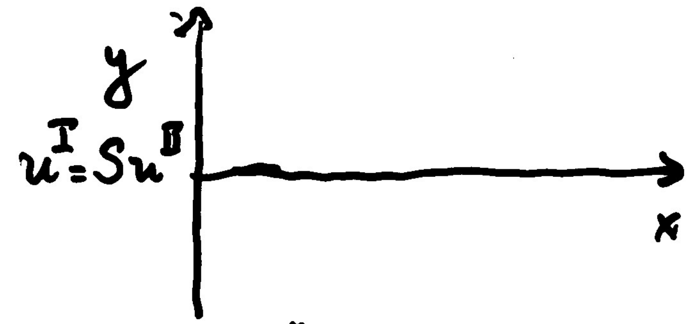
Let
$$\hat{u}_t = i|\omega|\hat{v}$$

$$\begin{pmatrix}\hat{v}\\ \hat{u}\end{pmatrix}_t = i|\omega| \begin{pmatrix}\hat{P}_1(\omega') & \hat{P}_0(\omega')\\ I & 0\end{pmatrix}\begin{pmatrix}\hat{v}\\ \hat{u}\end{pmatrix}$$

lower order terms
$$Qu = \sum B_j D_j u + B_0 u_t + Cu$$

2. Halfplan problems. (Cauchy problem is well posed

$$u_t = Au_x + Bu_y \quad , x \geq 0, t \geq 0, -\infty < y < \infty \tag{3}$$
$$u(x,y,0) = f(x,y)$$



$$A = \begin{pmatrix} -A^I & 0 \\ 0 & A^{II}\end{pmatrix} \quad \begin{matrix} A^I > 0 \\ A^{II} > 0\end{matrix}$$

$A = A^*$ if also $B = B^*$ and if the boundary conditions are dissipative then the problem is well posed in the same sence as before.

Constant coefficients.

<u>Lemma</u> The problem is not well posed if we can find a solution
$$u = e^{st + i\omega y}\hat{u}(x), \quad |\hat{u}|_\infty < \infty, \quad \mathrm{Re}\, s > 0$$
$$\hat{u}^I(0) = S\hat{u}^{II}(0)$$

$$s\hat{u} = A\hat{u}_x + i\omega B\hat{u}$$

$$\hat{u}_x = A^{-1}(sI - i\omega B)\hat{u} =: M\hat{u}$$

For $\mathrm{Re}\, s > 0$, there are no eigenvalues $\varkappa$ of M with $\mathrm{Re}\,\varkappa = 0$. There are exactly as many $\varkappa$ with $\mathrm{Re}\,\varkappa < 0$ as the number of boundary conditions.

General bounded solution

$$\hat{u} = \sum_{\mathrm{Re}\varkappa_j < 0} \sigma_j e^{\varkappa_j x} \underline{e}_j \qquad (4)$$

Introduce (4) into the boundary condition

$$C(s,\omega)\sigma = 0 \qquad (5)$$

<u>Theorem 3</u>. The problem is not well posed if (5) has a solution for some s, ω, $\mathrm{Re}\, s > 0$. The problem is well posed ~~is well posed~~ if (5) has no solution for $\mathrm{Re}\, s \geq 0$.

Well posed in what sense?

$$u_t = Au_x + Bu_y + F \qquad (6)$$

$$u(x,y,0) = 0, \quad u^{I}(0,y,t) = Su^{II}(0,y,t)$$

Laplace - Fourier transform

$$(s\hat{u} - A\hat{u}_x - iB\omega)\hat{u} = \hat{F} \qquad (7)$$

$$\hat{u}^{I}(0) = S\hat{u}^{II}(0)$$

If (5) has no solution then (7) has a unique solution for $s = i\xi + \eta$, $\eta > 0$

$$\|\hat{u}(\cdot,\omega,s)\| \leq K(\eta)\,\|\hat{F}(\cdot,\omega,s)\|$$

$$\int_0^\infty e^{-2\eta t}\,\|u(\cdot,\cdot,t)\|^2\,dt \leq K(\eta)\int_0^\infty e^{-2\eta t}\,\|F(\cdot,\cdot,t)\|^2\,dt$$

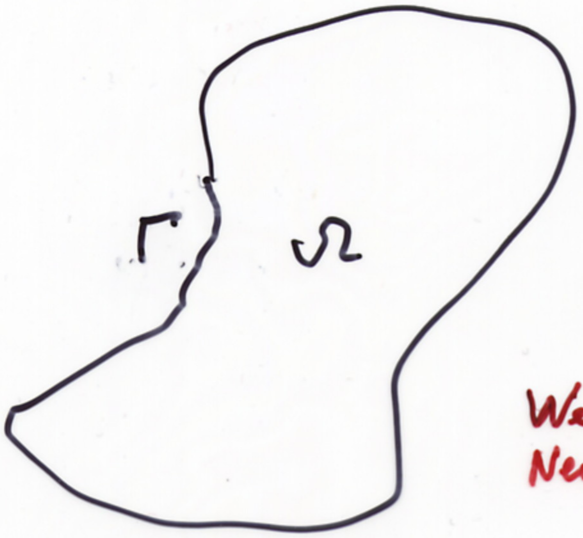
$$\lim_{\eta\to\infty} K(\eta) = 0$$

(Assumption: (3) is strongly hyperbolic and the multiplicity of the eigenvalues of $A\omega_1' + B\omega_2'$ is constant)
Lower order terms, frozen coefficients.

Work with A. Petersson and J. Ystrom

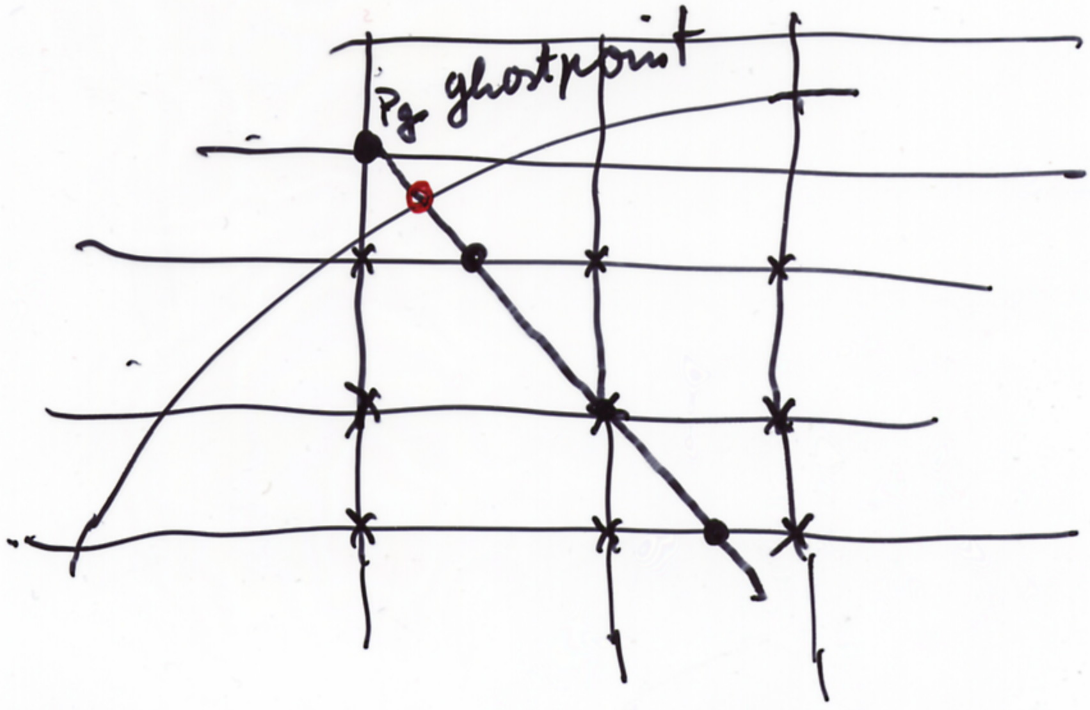
3) Difference approximations for the wave equation.

$$u_{tt} = \Delta u =: u_{xx} + u_{yy}$$
$$u(\underline{x},0) = f_1 \quad u_t(\underline{x},0) = f_2$$



$u = 0$
or
$\partial u/\partial n = 0$ on Γ

We discuss only Neuman conditions.



Relations between u at P_g and interior points approximate the boundary condition

Main challange : Stability

Main problem : Net is not alligned with the boundary (Some dissipation is needed).

Other methods : Finite element (variational principle) (unstructured meshes, implicite).

First order system ?

$$u_{tt} = u_{xx}$$

$$D_{+t} D_{-t} \tilde{u}(x,t) = D_{+x} D_{-x} \tilde{u}(x,t) \quad , (x,t) \text{ mesh point}$$

$$\begin{aligned} u_t &= v_x \\ v_t &= u_x \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \tilde{u}_t &= D_{0x} \tilde{v} \\ \tilde{v}_t &= D_{0x} \tilde{u} \end{aligned}$$

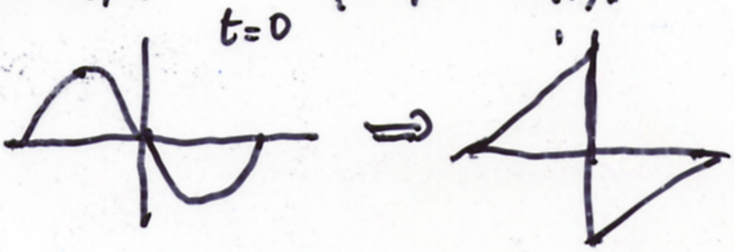
spurious waves.

$$u_t = u_x \implies \tilde{u}_t(x_\nu, t) = \frac{\tilde{u}(x_{\nu+1}, t) - \tilde{u}(x_{\nu-1}, t)}{2\Delta x}$$

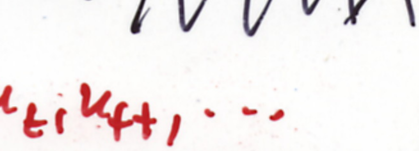
$$\tilde{u}(x_\nu, t) = (-1)^\nu \tilde{\tilde{u}}(x_\nu, t)$$

$$\tilde{\tilde{u}}_t(x_\nu, t) = -D_0 \tilde{\tilde{u}}(x_\nu, t) \quad (u_t = -u_x).$$

$$u_t + (u^2)_x = 0$$



$$\tilde{u}_t + D_0 u^2 = 0$$

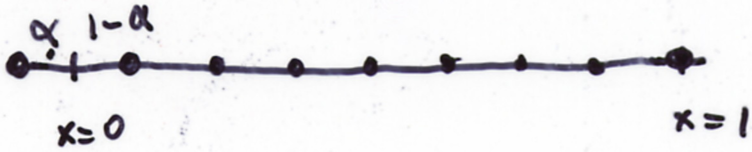


Smooth start ⟹ Estimates for $u_t, u_{tt}, \ldots$

$$\Delta u = F \qquad F = u_{tt}$$

1D

$$u_{tt} = u_{xx} \qquad 0 \le x \le 1$$



$x_\nu = (\nu-\alpha) h$

$\tilde{u}_\nu(t) = \tilde{u}(x_\nu$

$$\tilde{u}_{tt} = D_{+x} D_{-x} u_\nu \qquad \nu = 1, \dots N-1$$

Dirichlet: $(1-\alpha)\tilde{u}_0 + \alpha \tilde{u}_1 + \beta(\tilde{u}_2 - 2\tilde{u}_1 + \tilde{u}_0) = 0$

Neumann: $D_+ \tilde{u}_0 + h(\alpha - \frac{1}{2}) D_+^2 \tilde{u}_0 = 0$

$$\underline{\tilde{u}} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \end{pmatrix} \qquad \underline{\tilde{u}}_{tt} = \begin{pmatrix} a & b & 0 & \dots & 0 & \\ 1 & -2 & 1 & 0 & \dots & \\ 0 & 1 & -2 & 1 & 0 & \dots \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \dots & & \dots & \end{pmatrix} \underline{\tilde{u}}$$

Dirichlet $\quad a = -\dfrac{2-\alpha}{1-\alpha+\beta} \qquad b = \dfrac{1-\alpha}{1-\alpha+\beta}, \quad \beta \sim \frac{1}{2}$

Neumann $\quad a = -\dfrac{1}{\frac{3}{2}-\alpha}, \quad b = -a$

2D

$$u_{tt} = \Delta u$$

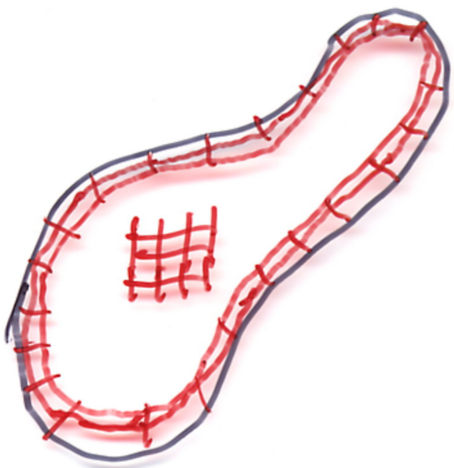
mesh alligned with the boundary

$$\tilde{u}_{tt} = \Delta_h \tilde{u}$$

$$\hat{\tilde{u}}_{tt} = D_{+x} D_{-x} \hat{\tilde{u}} - \frac{4}{(h_\cdot)^2} \sin^2\!\left(\frac{\omega h_\cdot}{2}\right) \hat{\tilde{u}}$$

No problem.

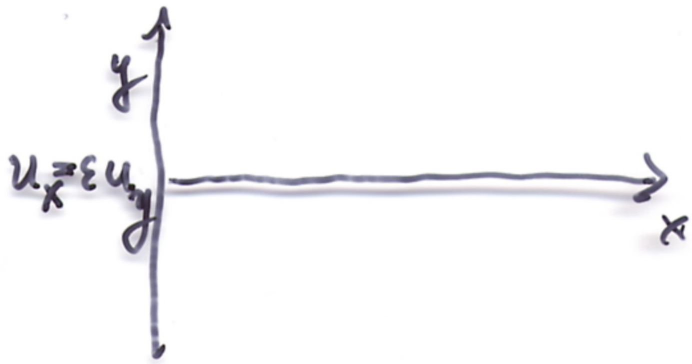
overlapping mesh



Discussion why there are problems if the mesh is not aligned with boundary.

Halfplan problem

$$u_{tt} = \Delta u$$



No energy estimate

$$u = e^{st + i\omega y}\,\hat{u}(x) \qquad \mathrm{Re}\, s > 0$$

$$\hat{u}_{xx} = (\omega^2 + s^2)\hat{u}, \quad \text{for } x \geq 0$$

$$\hat{u}_x = i\omega\hat{u} \quad \text{for } x = 0$$

$$\hat{u} = \sigma_1 e^{-\sqrt{\omega^2+s^2}\,x} + \sigma_2 e^{\sqrt{\omega^2+s^2}\,x} \qquad (1)$$

$$\sigma_2 = 0, \quad -\sigma_1\left(\sqrt{\omega^2+s^2} - i\omega\right) = 0 \implies \sigma_1 = 0$$

There are no solutions for Real $s > 0$.

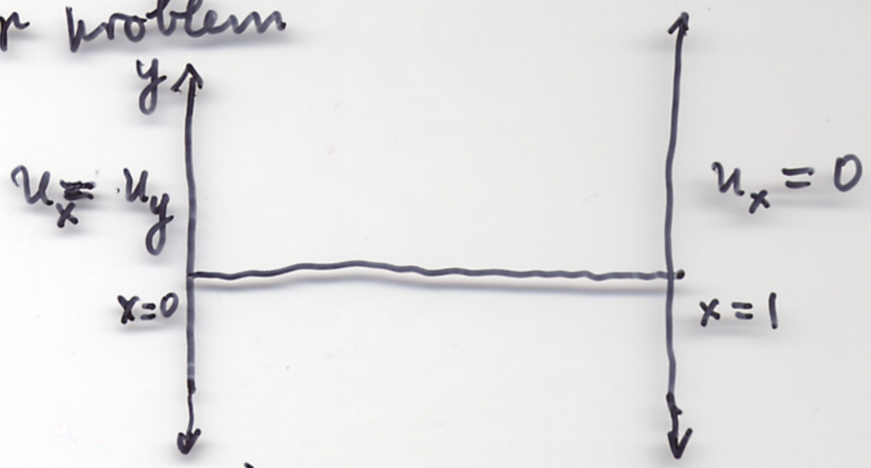
There are solutions

$$\hat{u} = \sigma_1 e^{-\sqrt{\omega^2+s^2}\,x} \quad \text{for } \operatorname{Re} s = 0$$

$$s = i\omega\sqrt{1+\varepsilon}\,, \quad \sqrt{\omega^2+s^2} = \varepsilon i\omega$$

Highly oscillatory for large ω

Strip problem



$$u = e^{st+i\omega y}\,\hat{u}(x)\,, \quad \hat{u}(x) \text{ given by } (1)$$

$$\operatorname{Re} s \approx \frac{1}{\sqrt{8}} \log(2|\omega|)\,, \quad e^{(\operatorname{Re} s)t} = (2|\omega|)^{t/\sqrt{8}}$$

Highly oscillatory instability can be controlled by adding a dissipative term to the differential equation.

$$u_{tt} = \Delta u + 2a\,u_{xy} \qquad |a| < 1$$

$\alpha = a$ is the only well posed problem

$$u_{tt} = \Delta u, \quad x \geq 0$$

$$u_x = \alpha u_{yy} \quad \text{at } x = 0$$

$$u = e^{st + i\omega y}\,\hat{u}(x), \quad \operatorname{Re} s > 0$$

$$\hat{u}(x) = e^{-\sqrt{\omega^2 + s^2}\,x},$$

$$-\sqrt{\omega^2 + s^2} = -\alpha\omega^2$$

1) If $\alpha < 0$ there are no solutions for $\operatorname{Re} s \geq 0$.

2) If $\alpha > 0$ then $s \sim \alpha\omega^2$, $-\sqrt{\omega^2 + s^2} \sim \alpha\omega^2$, $|\omega| \gg 1$

Boundary layer instability cannot be controlled by dissipative terms in the differential equation.

Modified equation

$(*)\qquad \tilde{u}_t + D_0 \tilde{u} = \nu D_+ D_- \tilde{u} \qquad\qquad u_t + u_x = \nu u_{xx}$

$$u_t + u_x + \frac{1}{6} h^2 u_{xxx} = \nu u_{xx} + \frac{\nu}{12} h^2 u_{xxxx} + O(h^4)$$

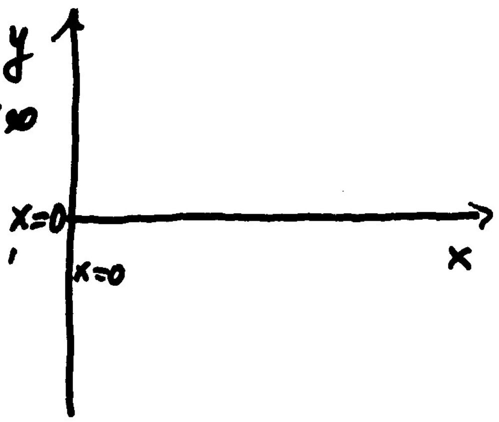
$(**)\qquad w_t + w_x + \frac{1}{6} h^2 w_{xxx} = \nu w_{xx} \qquad\qquad |\omega h| \ll 1$

We use $(**)$ to discuss the behavior of $(*)$

Modified problem

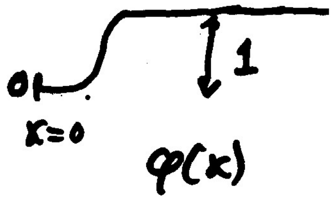
$$u_{tt} = \Delta u \qquad x \geq 0,\ -\infty < y < \infty$$

$$u_x = \beta_1 h^2 u_{yyy} + \beta_2 h^3 u_{yyyy}, \quad x = 0$$



$$u_{tt} = \Delta u - \alpha h^3 \Delta(\varphi(x) \Delta u_t)$$

$$u_x = \beta_1 h^2 u_{yyy} + \beta_2 h^3 u_{yyyy} + \gamma h^3 u_{yyyy},$$



$\gamma_2 > 0$, $\alpha > 0$ sufficiently large.

First [illegible] dissipation terms [illegible] and for the [illegible] tangential differences. Terms added to [illegible]

Code is very robust. The amount of dissipation is very small and one can calculate for long times. However, if there are corners we had to make ad hoc decisions how to calculate tangential derivatives.

We eliminate the ghostpoints

$$\Delta_h u \longrightarrow A\underline{u}$$

$$\underline{u}_{tt} = A\underline{u} - \alpha h^3 A^* A \underline{u}_t$$

(Away from the boundary $A = \Delta_h$, $A^*A = \Delta_h^2$)

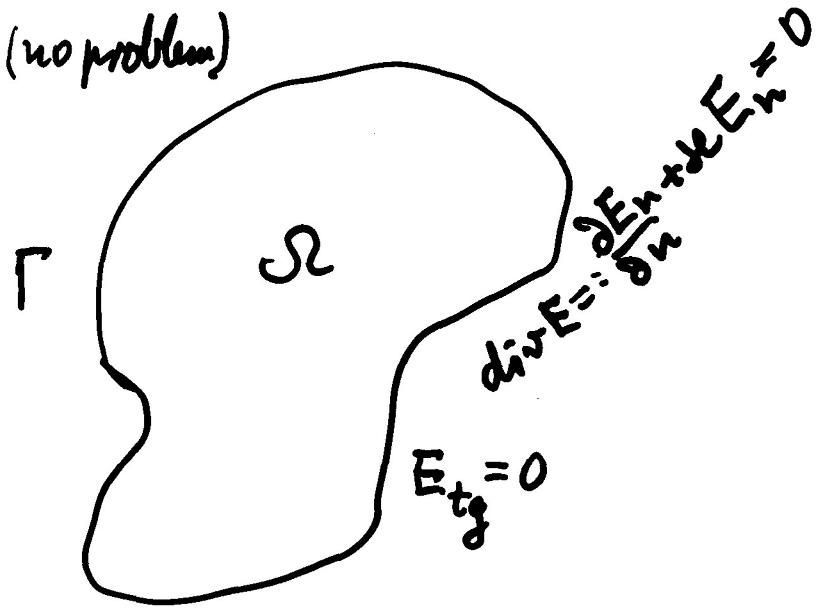
$$D_{+t} D_{-t} \underline{u}_t = A\underline{u} - \alpha h^3 A^* A D_{-t} \underline{u}$$

Higher order method away from the boundary.

Maxwell's equations.

$$E_{tt} = \Delta E, \quad E = \begin{pmatrix} E^{(x)} \\ E^{(y)} \end{pmatrix}$$

$\operatorname{div} E = 0$ (no problem)



Elastic wave equation. Model problem (halfplan)

$$u_{tt} = \Delta u + 2a u_{xy} \qquad |a| < 1,\ x \geq 0,$$

$$u_x + \alpha u_y = 0 \quad x = 0$$

only well posed if $\alpha = a$

$$u_{tt} = 2a u_{xt} + b u_{xx}, \quad b > -a^2.$$

$$u_{tt} = 2a u_{xt} + b u_{xx}$$