Critical Collapse in a General Newtonian Self-Gravitating System

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Critical collapse phenomena in Newtonian gravity will be investigated. The gravitational field will be modelled using a massive complex scalar field, and in the Newtonian limit the equations of motion reduce to a coupled system of Schrödinger-Poisson equations. Critical collapse will be looked for numerically in a 1-parameter family of initial data, and if found will lead to investigations of self-similarity of solutions. The equations of motion will be approximated using a Crank-Nicholson finite-differencing scheme and numerical simulations will be run using ADI and multigrid methods. If self-similarity is found there will be a need to implement some form of mesh refinement to follow the evolution over a wide range of length-time scales.

I. MOTIVATION

Singular solutions in gravitational systems have been of much interest since the discovery of the Schwarzchild solution of a spherically symmetric black hole in General Relativity in 1917. In the 1960s, singularity theorems were developed which guarantee independently of symmetry or type of mass the existence of a black hole[1]. In the early 1990s, critical collapse of self-gravitating systems was first observed by Choptuik in numerical studies of a massless scalar field in spherical symmetry[2], and describes systems which are near the threshold between formation of a singularity in gravitational collapse and the total dispersion of the mass.

For precisely fine tuned initial conditions, the system may tend towards a critical solution between singularity formation and dispersion. Numerical studies[2][3][4] of self-gravitating systems show that critical solutions exhibit either time independence or self-similarity (for a review of self-similar solutions in General Relativity, see [5]), and are labeled Type I and Type II critical phenomena, respectively. They are so labelled because of their resemblence to Type I and Type II phase transistion in statistical mechanics. For Type I critical phenomena the mass of the stable singular solution is a certain fraction of the mass of the critical solution, but for Type II the mass of the stable singular solution obeys a power scaling law, and initial conditions can be set to make the resulting black hole mass infinitesimally small.

Self-similarity has also been found in the stable singular solutions to a self-gravitating perfect gas in general relativity[6], and thus form from generic initial conditions. This provides support for the self-similarity hypothesis which states that self-similarity is a generic phenomenon in critical collapse. These self-similar solutions are also known to have naked singularities, and thus provide a strong counter-example for the cosmic censorship conjecture (see [7] for a review on cosmic censorship). It is generally believed that some form of cosmic censorship is valid, although the precise form has still yet to be defined, and so critical collapse offers a way of testing which forms of cosmic censorship might be valid. Furthermore, the range of validity of the self-similarity hypothesis has not been tested significantly beyond spherical symmetry, so whether or not self-similarity is a generic feature of final states in critical gravitational collapse is still an open question.

Investigations of critical collapse in purely Newtonian self-gravitating systems have been carried out for a Newtonian isothermal gas[8], and also scalar fields with or without mass/self-interaction potential. These systems also exhibit self-similar critical and stable solutions which have naked singularities, and so provide another window for investigating the range of validity of cosmic censorship. Comparatively less work has been done on Newtonian systems than on general relativistic ones; however, due to the reduced complexity associated with Newtonian gravity, it offers a good laboratory for investigation of critical collapse under deviations from spherical symmetry and other simplifications that are common place in general relativistic investigations, and therefore a means of probing the validity of cosmic censorship and the self-similarity hypothesis. It should be noted that since these analyses are purely Newtonian, they could potentially have been carried out long before the development of general relativity. There is presently no explanation for the fact that stable solutions exhibit self-similarity in terms of dynamics, so an understanding of the Newtonian case may provide insight into the role that the dynamics play in creating self-similar solutions.

Here we are interested in investigating critical collapse in Newtonian self-gravitating systems which develop so called "blow-up" solutions, ie. solutions which become singular in finite time. Such solutions are interesting for a number of reasons. As mentioned above, investigating the evolution of solutions as they become singular will provide insight into how singular solutions form out of the dynamics. Blow-up solutions are also interesting because in the context of Newtonian gravity, it is not clear that critical phenomena will occur for any matter model. In general relativity, the universality of coupling of mass-energy-momentum guarantees that there will be black

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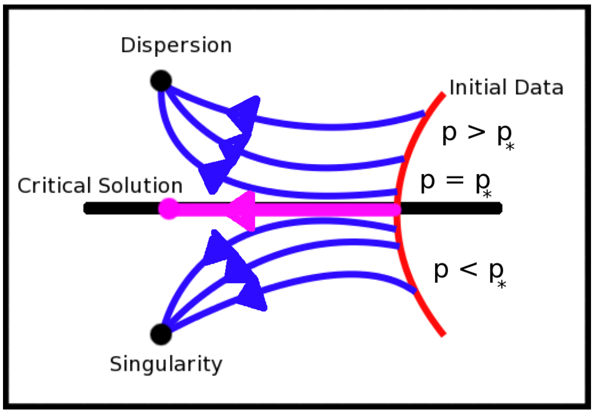

FIG. 1: The phase space portrait near a critical point. The red line represents a 1-parameter family of initial gravity configurations, and the blue lines extending from the initial data represent trajectories in the phase space of possible Z. The red line is drawn to cross the singularity threshold at $p = p_*$, and for this value of p the gravity configuration tends towards the critical solution.

hole formation in any matter model; however, Newtonian gravity only sets a universality of coupling of mass to momentum, and it is not clear that all matter models will admit initial data that result in blow-up.

II. THEORY

A. Properties of Critical Collapse

We now describe in greater detail some of the properties of critical collapse elluded to in the motivation section. The mathematical ideas sketched out here are modelled from Gundlach[9].

1. *Type I Critical Phenomena*

In investigating critical collapse, one is concerned with a function Z which represents the strength of the gravitational field, and is in general a function of both space and time, $Z = Z(t, r)$. Z can also depend on any number of parameters depending on the physical system in question. In the phase space of parameters, there will be a surface defining initial Z which will tend towards the critical solution. Varying a single parameter, p, holding all others fixed will produce initial configurations Z which generically will have an intersection with this surface. Fig.1 shows a one parameter family of initial configurations, and their resulting "trajectories" in phase space, ultimately resulting in one of three possible outcomes: singularity formation, total dispersion, or a critical solution.

The critical solution in a Type I critical phenomenon is observed to be either independent of time or period in time. Here we focus on the case where the solution is independent of time, and so we can write $Z_* = Z_*(r)$, and its general linear perturbation can be written as

$$Z(r,t) = Z_*(r) + \sum_i C_i(p) e^{\lambda_i t} Z_i(r) \tag{1}$$

numerical simulations suggest that when looking at a one parameter family of initial configurations, there is only one growing mode ($\lambda_0 > 0$), and so for Z close to the critical solution we have the approximation

$$Z \simeq Z_*(r) + \frac{dC_0}{dp}(p_*)(p - p_*) e^{\lambda_0 t} Z_0(r) \tag{2}$$

Initial configurations Z will all approach the critical solution before the growing mode forces them into one of the two stable configurations. The time spent near the critical solution will be longer for initial configurations with $p - p_*$ closer to zero. This can be made quantitative in defining a time, t_p by

$$\frac{dC_0}{dp} |p - p_*| e^{\lambda_0 t_p} \equiv \epsilon \tag{3}$$

where ϵ is an arbitrarily small positive constant. By setting ϵ to a particular small value this causes t_p to be interpretted as the time at which Z satisfies

$$Z(r, t_p) \simeq Z_*(r) \pm \epsilon Z_0(r) \tag{4}$$

where $Z_0(r)$ is the initial configuration of gravity, and the sign in front of ϵ is the sign of $(p - p_*)$. From its definition we can see that t_p scales as

$$t_p = -\frac{1}{\lambda_0} ln|p - p_*| + const \tag{5}$$

t_p can be thought of as the time up to which the solution $Z(t, r)$ is significantly dependent on $(p - p_*)$. After $t = t_p$ the solution will quickly converge to a stable solution and the information of the magnitude of $(p - p_*)$ is lost.

It is also a matter of observation through numerical simulations that the mass of the stable singular solution is a certain fraction of the mass of the critical solution, *independent* of initial data. Therefore, Type I critical collapse usually occurs in physical systems where a mass scale is set.

2. *Type II Critical Phenomena*

Type II critical phenomena are observed to have critical solutions which exhibit scale-invariance, or self-similarity. This symmetry comes in a continuous and discrete form, as it did in Type I phenomena, and to facilitate this symmetry we need to introduce new variables, for example

$$x = -\frac{r}{t}, \quad \tau = -ln(-\frac{t}{l}), \quad t < 0 \tag{6}$$

where the origin of t has been moved so $t = 0$ and $\tau \to \infty$ corresponds to the time of critical collapse in the critical case, and l is a dimensionful parameter with units length which depends on the 1-parameter family of initial data.

If we now define gravity configurations by $Z = Z(\tau, x)$, and the critical solution is $Z_*(\tau, x)$, then we can see that $Z_*(\tau, x)$ is continuously scale-invariant if it is independent of τ, and this property is called continuous self-similarity (CSS). The discrete version is when $Z_*(\tau, x)$ is periodic in τ with

$$Z_*(\tau, x) = Z_*(\tau + \Delta, x) \tag{7}$$

and this is called discrete self-similarity (DSS), because in going from τ to $\tau+\Delta$, the solution is the same but the space and time scales (x and t) are smaller by a factor of $e^{-\Delta}$.

Working through the same perturbation analysis as with Type I, we arrive at an analogous relation for time scaling

$$\tau_p = \frac{1}{\lambda_0} ln|p - p_*| + const \tag{8}$$

Where τ represents a measure of the time spent near the critical solution. In analogy with Eq.4, and shifting the origin of t to correspond with $\tau = \tau_p$, the solution can be written as

$$Z(0, r) = Z_*(-\frac{r}{L_p}) \pm \epsilon Z_0(-\frac{r}{L_p}) \tag{9}$$

where $L_p = le^{-\tau_p}$, which now represents an overall scale.

Now, in the units we're working in, $c = G = 1$, then mass also has units of length, which means the mass of the singularity is proportional to L_p

$$M \propto L_p \propto (p - p_*)^{1/\lambda_0} \tag{10}$$

where the last proportionality follows from the definition of τ, Eq.8, and we have found the so called critical exponent $\gamma = 1/\lambda_0$

3. Equations of Motion

The discussion here on the equations of motion and finite differencing schemes is heavily influenced by Choi[10]. In the limit of Newtonian gravity, ie., weak gravitational fields and velocities much smaller than the speed of light, the complex massive scalar field, Ψ, with Newtonian gravitational potential, V, satisfies a Schrödinger equation coupled to a Poisson equation

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\nabla^2\Psi + V\Psi \tag{11}$$

$$\nabla^2 V = \Psi\Psi^* \tag{12}$$

where we are working in units with $G = c = \hbar = 1$, and the physical quantities are given by

$$\mathbf{x}_{phy} = \frac{\hbar}{mc}\mathbf{x}_{comp} \tag{13}$$

$$t_{phy} = \frac{\hbar}{mc^2} t_{comp} \tag{14}$$

$$\Psi_{phy} = \frac{c^2}{\sqrt{4\pi G}}\Psi_{comp} \tag{15}$$

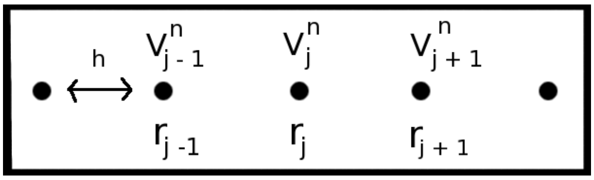

FIG. 2: Discretization of r-axis for finite-differencing.

III. DETAILS ON PROPOSED CALCULATION

To solve Eq.11 and Eq.12 numerically, a finite differencing scheme must be employed. A popular choice for numerically solving Schrödinger type equations is to use a Crank-Nicolson (CN) scheme, but the Poisson equation can be solved using simpler techniques.

In spherical symmetry, the laplacian takes the form

$$\nabla^2 = \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} \tag{16}$$

A basic discretization of the r-axis is shown in figure 2. To solve Eq.12 on this mesh, the differential operators need to be replaced by finite-difference operators defined on this mesh, and V and $\Psi\Psi^*$ replaced by vectors with components of V and $\Psi\Psi^*$ on the mesh points. The standard approximation is

$$\frac{\partial V}{\partial r}(r) \simeq \frac{V(r+h) - V(r-h)}{2h} \tag{17}$$

$$\frac{\partial^2 V}{\partial r^2}(r) \simeq \frac{V(r+h) - 2V(r) + V(r-h)}{h^2} \tag{18}$$

Eq. 12 can now be written as a matrix equation

$$L\mathbf{V} = \mathbf{\Psi\Psi}^* \tag{19}$$

where L has the form

$$L = \frac{1}{h^2} * \begin{pmatrix} h^2 & 0 & 0 & 0 & 0 \\ \frac{(r_2-h)}{r_2} & \frac{-2r_2}{r_2} & \frac{(r_2+h)}{r_2} & 0 & 0 \\ 0 & \frac{(r_3-h)}{r_3} & \frac{-2r_3}{r_3} & \frac{(r_3+h)}{r_3} & 0 \\ 0 & 0 & \frac{(r_4-h)}{r_4} & \frac{-2r_4}{r_4} & \frac{(r_4+h)}{r_4} \\ 0 & 0 & 0 & 0 & h^2 \end{pmatrix}$$

Eq.19 is then inverted to solve for V, and plugged into equation 11, and now we have a PDE for Ψ. A CN scheme can then be implemented to solve for Ψ on the computational domain, which is documented elsewhere.

At this point in time, the research does not call upon any new numerical techniques, and all the techniques required are well documented in the literature of numerical relativity. Other hardware and software requirements are listed below.

IV. RESOURCES LIST

Preliminary work on critical collapse using complex massive scalar fields in spherical symmetry will only require the use of a single personal computer, however if singular solutions or self-similarity are found the computations will become more expensive. The UBC physics dept. has two computer clusters available for computationally expensive calculations. The calculations will require the use of software developed by Choptuik for the visualization of time dependent PDEs, namely XVS and RNPL.

V. PLANNED SCHEDULE

Development of initial code, and preliminary simulations should be completed by the beginning of January 2009. If self-similarity is found, exact construction of the self-similar solutions and critical exponent for mass scaling may be pursued during the month of January. More expensive calculuations could be pursued from February to April, including critical collapse simulations in axialsymmetry or other symmetry, or investigations using 2-parameter initial data. Writing of the thesis should begin at the beginning of March.

VI. ACKNOWLEDGEMENTS

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