Isolated and Dynamical Horizons

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Outline

- Introduction
- Basic Notions
  - Isolated Horizons
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- Application to Numerical Relativity
- Conclusion

Introduction

- What is the black hole (BH) research scope?
  - Since the seventies, black holes have played an important role in quantum gravity, mathematical physics, numerical relativity and gravitational wave phenomenology.
  - Its mathematical theory turns out to be extremely rich and full of surprises:
  - Mathematical General Relativity: Laws of black hole mechanics; black hole uniqueness theorems; analysis of the detailed properties of Kerr-Newman solutions.
  - Numerical Relativity: BH formation through gravitational collapse; critical phenomenon associated; dynamics leading to their coalescence.
  - Relativistic Astrophysics: the most promising sources of gravitational waves.
- Is there a unified mathematical framework?
  - Despite all the progress made in these fields, the conceptual settings and mathematical models used to discuss them have varied considerably from one area to another.
  - To analyze diverse facets of black holes in a unified manner, quasi-local framework was introduced over the last five years.
  - In this framework, evolving black holes are modelled by dynamical horizons and black holes in equilibrium by isolated horizons.

Introduction

- Dynamical Situations:

- For fully dynamical black holes, there is one major result in exact general relativity:

  - Area theorem: the area of the black hole event horizon can never decrease, if matter satisfies the null energy condition ($R_{ab}W^aW^b \geq 0$).

$$\delta a \geq 0 \tag{1}$$

  - Note that it is a qualitative result since it doesn't provide a formula for the amount by which the area increases in physical processes.
  - First law of black hole mechanics: For a black hole of ADM mass M, angular momentum J, area a, surface gravity κ, and angular velocity Ω,

$$\delta M = \frac{\kappa}{8\pi G}\delta a + \Omega\delta J \tag{2}$$

  - Note that it does relate the change in the horizon area to that in the energy and angular momentum, as the black hole makes a transition from one equilibrium state to a nearby one.

Introduction

- Dynamical Situations:

- Is there a full dynamical version of the first law? Need to quantitatively control how black holes grow in exact general relativity.

- Note however that the event horizons can form and grow even in a flat region of space-time.

- In the standard framework where the surface of the black hole is represented by an event horizon, it is impossible to obtain the desired formula.

- Equilibrium Situations:

- Black holes in equilibrium (isolated BH) are usually represented by stationary solutions of field equations, i.e, solutions which admit a time-translational Killing vector field *everywhere*.

- Global stationarity is too restrictive: Physically, we want to impose boundary conditions on the horizon such that *only* the black hole itself is isolated, i.e. demand only that the intrinsic geometry of the horizon be time independent.

Introduction

- Equilibrium Situations:

- In realistic situations, one is typically interested in the final stages of collapse where the black hole has formed and 'settled down'.

- There is likely to be gravitational radiation and non-stationary matter far away from the black hole.

- Another unsatisfactory feature comes from the first law of BH mechanics: the angular momentum J and the mass M are defined at infinity while the angular velocity Ω and surface gravity κ are defined at the horizon.

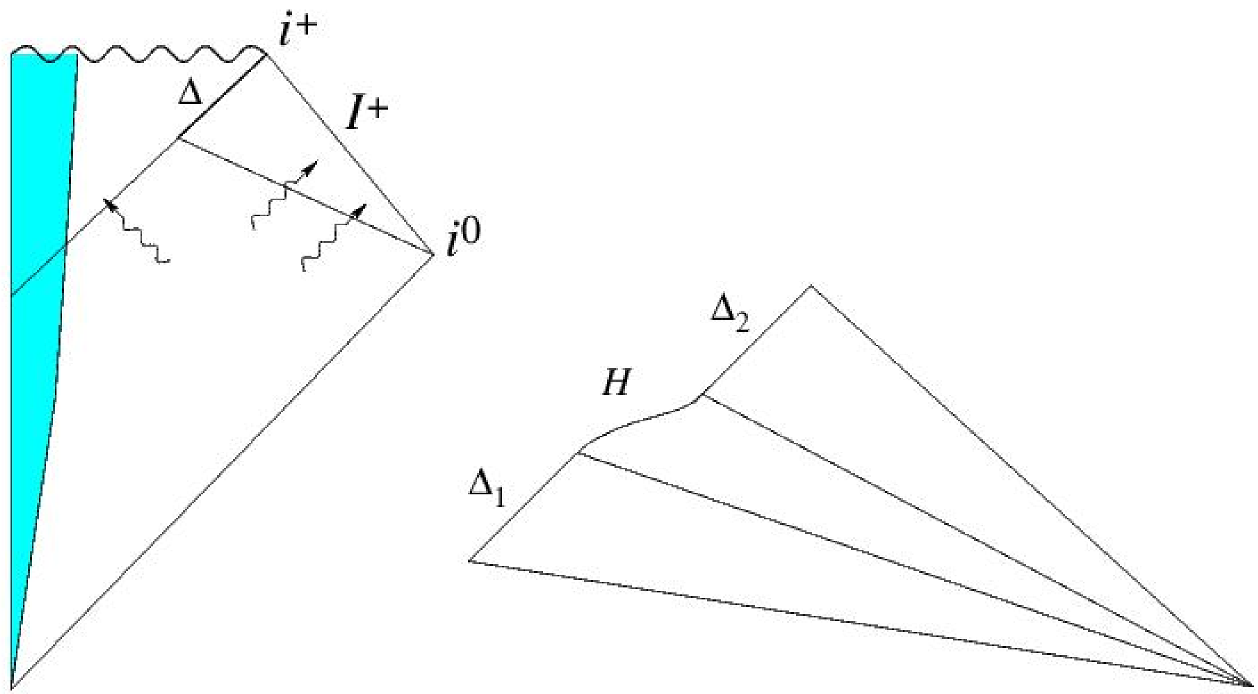
- This back and forth between the horizon and infinity raises a question:

  - If there is matter around the BH that contributes for the angular momentum and mass at infinity, should this contribution be relevant for the first law?

- Therefore, in order to make the meaning of the first law more transparent, a more suitable paradigm to replace frameworks based on event horizons in stationary space-times is desired.

Introduction

- Penrose diagram for a typical gravitational collapse:



Left diagram: The portion Δ of the event horizon at late times is isolated. One would expect the first law to apply to Δ even though the entire space-time is not stationary.

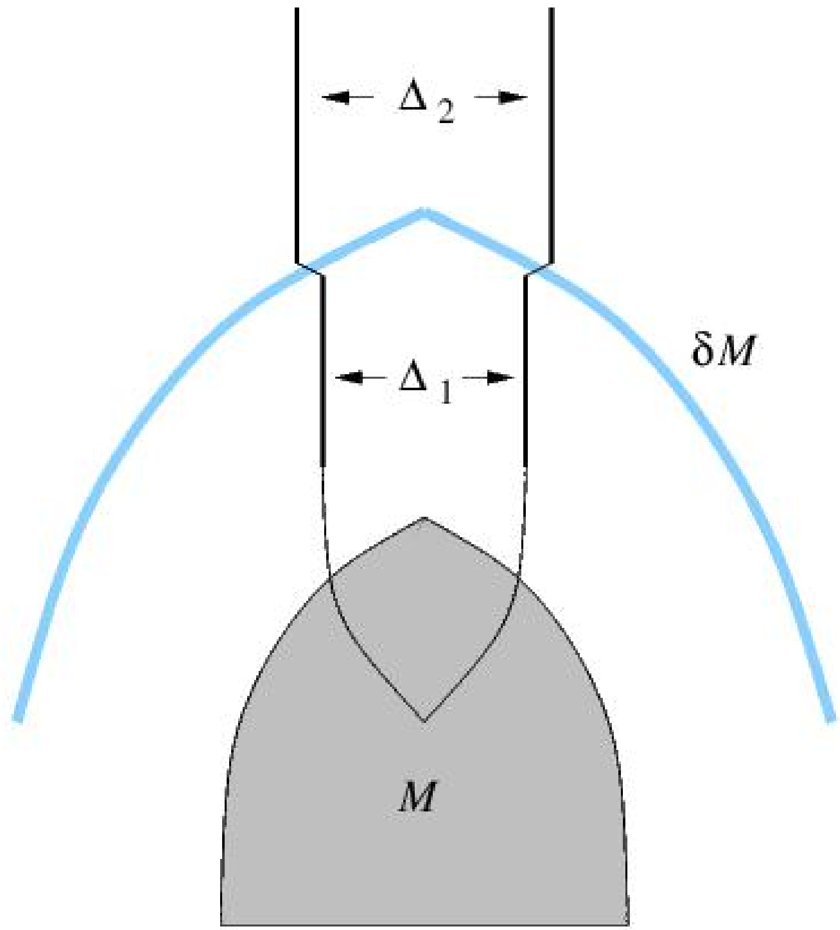
Right diagram: a black hole initially in equilibrium, absorbs a finite amount of radiation, and again settles down to equilibrium.

Introduction

- Global Nature of event horizons:
- The future event horizon is defined as the future boundary of the causal past of future null infinity.
- Problem with this definition: It is too global for many applications.
- We can only speak of a black hole after we have constructed the entire space-time.
- Thus, for example, an event horizon may well be developing in the room you are now sitting in anticipation of a gravitational collapse that may occur in this region of our galaxy a million years from now.
- However, when astrophysicists say that they have discovered a black hole in the center of our galaxy, they are referring to something much more concrete and quasi-local than an event horizon.

Introduction

- Global Nature of event horizons:



A spherical star of mass M undergoes collapse. Much later, a spherical shell of mass δM falls into the resulting black hole. While Δ_1 and Δ_2 are both isolated horizons, only Δ_2 is part of the event horizon.

Isolated Horizons

- Raychaudhuri's equation: Describes the rate of change of the expansion θ along each timelike geodesics of a congruence:

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}\xi^a\xi^b \tag{3}$$

where the expansion θ, shear σ_{ab} and twist ω_{ab} are given by:

$$\theta = B^{ab}h_{ab} \tag{4}$$

$$\sigma_{ab} = B_{(ab)} - \frac{1}{3}\theta h_{ab} \tag{5}$$

$$\omega_{ab} = B_{[ab]} \tag{6}$$

and

$$B_{ab} = \nabla_b\xi_a \qquad\qquad h_{ab} = g_{ab} + \xi_a\xi_b \tag{7}$$

- For null geodesics, $k_a k^a = 0$

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}k^a k^b \tag{8}$$

Isolated Horizons

- The key idea is to extract from the notion of a Killing horizon the minimal conditions which are necessary to define physical quantities such as the mass and angular momentum of the black hole and to establish the zeroth and the first laws of black hole mechanics.

- Definition 1: A sub-manifold Δ of a space-time (M, g_{ab}) is said to be a non-expanding horizon (NEH) if

  1. Δ is topologically $S^2 \times R$ and null;
  2. any null normal ℓ^a of has vanishing expansion, $\theta_{(\ell)}$; and
  3. all equations of motion hold at Δ and the stress energy tensor T_{ab} is such that $-T^a_b \ell^b$ is future-causal for any future directed null normal ℓ^a.

- Note that the intrinsic metric $q_{ab} = \phi_* g_{ab}$ on Δ has signature $(0, +, +)$.

- Since q_{ab} is degenerate it has an inverse in a 'weaker' sense: $q_{am} q_{bn} q^{mn} = q_{ab}$

- It can be shown that condition 2 results in $\mathcal{L}_\ell q_{ab} = 0$ on Δ. Thus, the intrinsic metric on an NEH is 'time-independent'; this is the sense in which an NEH is in equilibrium.

Isolated Horizons

- Weakly isolated horizons are characterized by the property that, in addition to the metric q_{ab}, the connection component $\mathcal{D}_a \ell^b$ is also 'time independent'.

- Definition 2: The pair $(\Delta, [\ell])$ is said to constitute a weakly isolated horizon (WIH) provided Δ is an NEH and each null normal ℓ^a in $[\ell]$ satisfies

$$(\mathcal{L}_\ell \mathcal{D}_a - \mathcal{D}_a \mathcal{L}_\ell)\ell^b = 0 \tag{9}$$

- If the full connection $\mathcal{D}$ is time-independent:

- Definition 3:A WIH $(\Delta, [\ell])$ is said to constitute an isolated horizon (IH) if

$$(\mathcal{L}_\ell \mathcal{D}_a - \mathcal{D}_a \mathcal{L}_\ell)V^b = 0 \tag{10}$$

for arbitrary vector fields V^a tangential to Δ.

Dynamical Horizons

- Definition 4: A smooth, three-dimensional, space-like sub-manifold (possibly with boundary) H of space-time is said to be a dynamical horizon (DH) if it can be foliated by a family of closed 2-manifolds such that

  1. on each leaf S the expansion $\theta_{(\ell)}$ of one null normal ℓ^a vanishes; and
  2. the expansion $\theta_{(n)}$ of the other null normal n^a is negative.

- Note that dynamical horizons are not defined relative to a space-like surface as was the case with Hawking's apparent horizons nor do they make any reference to infinity as is the case with event horizons.

Dynamical Horizons

- Example: The 4-metric of the Vaidya space-time is given by:

$$g_{ab} = -\left(1 - \frac{2GM(v)}{r}\right) \nabla_a v \nabla_b v + 2\nabla_{(a} v \nabla_{b)} r + r^2 \left(\nabla_a \theta \nabla_b \theta + sin^2\theta \nabla_a \phi \nabla_b \phi\right) \tag{11}$$

  where $M(v)$ is any smooth, non-decreasing function of v and (v, r, θ, ϕ) are the ingoing Eddington-Finkelstein coordinates.

- It is a spherically symmetric solution to Einstein's equations with a null fluid as source

- This is a solution of Einstein's equations, the stress-energy tensor being given by

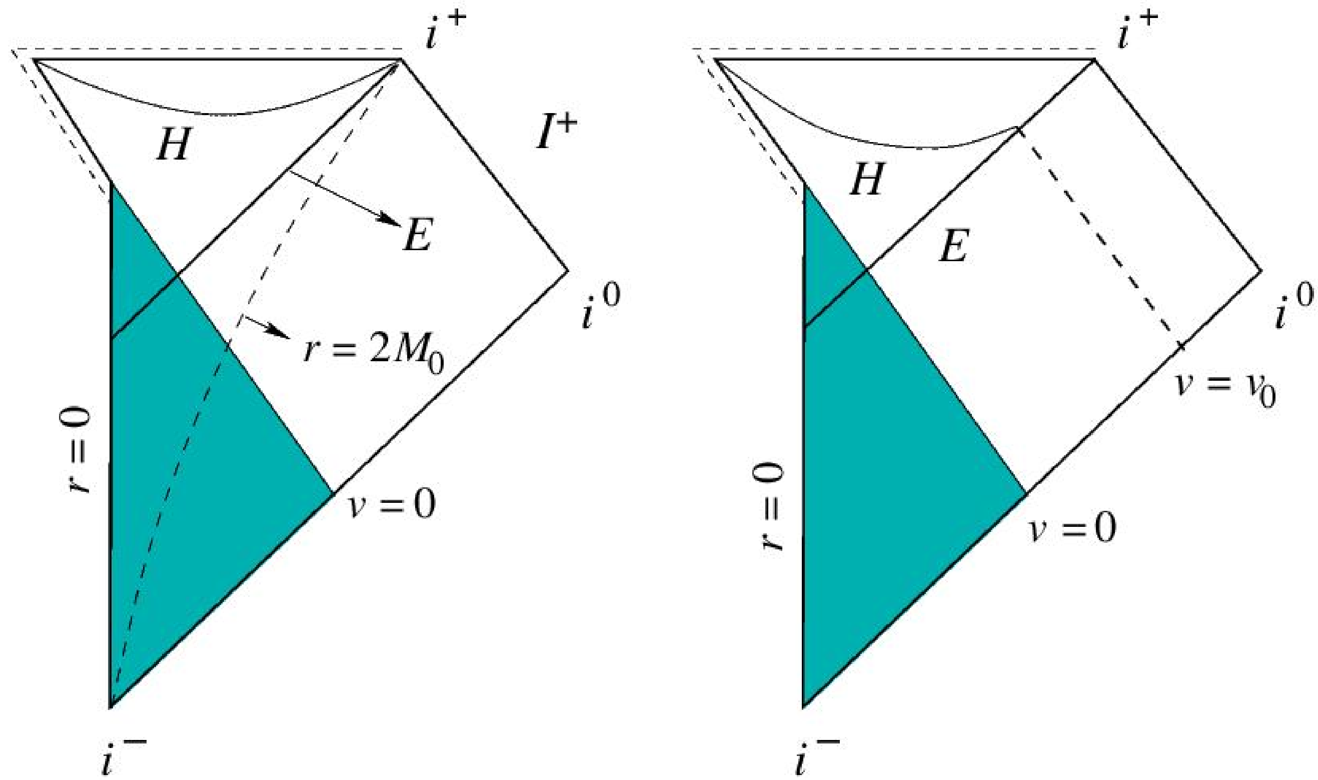
$$T_{ab} = \frac{\dot{M}(v)}{4\pi r^2} \nabla_a v \nabla_b v \tag{12}$$

  where $\dot{M} = dM/dv$.

- T_{ab} satisfies the dominant energy condition if $\dot{M} \geq 0$, and vanishes if and only if $\dot{M} = 0$.

Dynamical Horizons

- Example: Vaidya space-time:



Penrose diagrams of Schwarzschild-Vaidya metrics for which the mass function $M(v)$ vanishes for $v \leq 0$. In the left panel, as v tends to infinity, $\dot{M}$ vanishes and M tends to a constant value M_0.In both figures, the event horizon originates in the shaded flat region, while the dynamical horizon exists only in the curved region.

Applications in numerical relativity

- By its own nature, numerical relativity is intrinsically related to choices of coordinates, gauge conditions, dynamical variables, etc...
- That nature makes non-trivial to extract gauge invariant physics from the numerical simulations of spacetimes.
- Traditionally, the analytical infrastructure available for this purpose is based on properties of the Kerr solutions and its perturbations.
- Is it reliable to apply this 'intuition' in the fully dynamical, strong curvature regime?
- The IH and DH framework represents an 'upgrade' of this analytical infra-structure to extract physics more reliably and with greater accuracy.
- Examples:
  - Calculations of mass and angular momentum of black holes.
  - Initial data (ID): construction of quasi-equilibrium ID and calculation of the gravitational binding energy for a binary BH.
  - Calculation of source multipole moments for BHs.
  - Extraction of gauge invariant waveforms.

Applications in numerical relativity

- Numerical computation of black hole mass and angular momentum:

- The mechanics of IHs and DHs provides expressions of angular momentum and mass of the horizon:

$$J_\Delta = -\frac{1}{8\pi}\oint_S (\omega_a \varphi^a)^2 \epsilon \tag{13}$$

where ω_a is a 1-form on Δ such that, for a null vector ℓ^a the surface gravity is defined:

$$\kappa_{(\ell)} = \ell^a \omega_a \tag{14}$$

and ϵ_{ab} is the natural area 2-form on Δ.

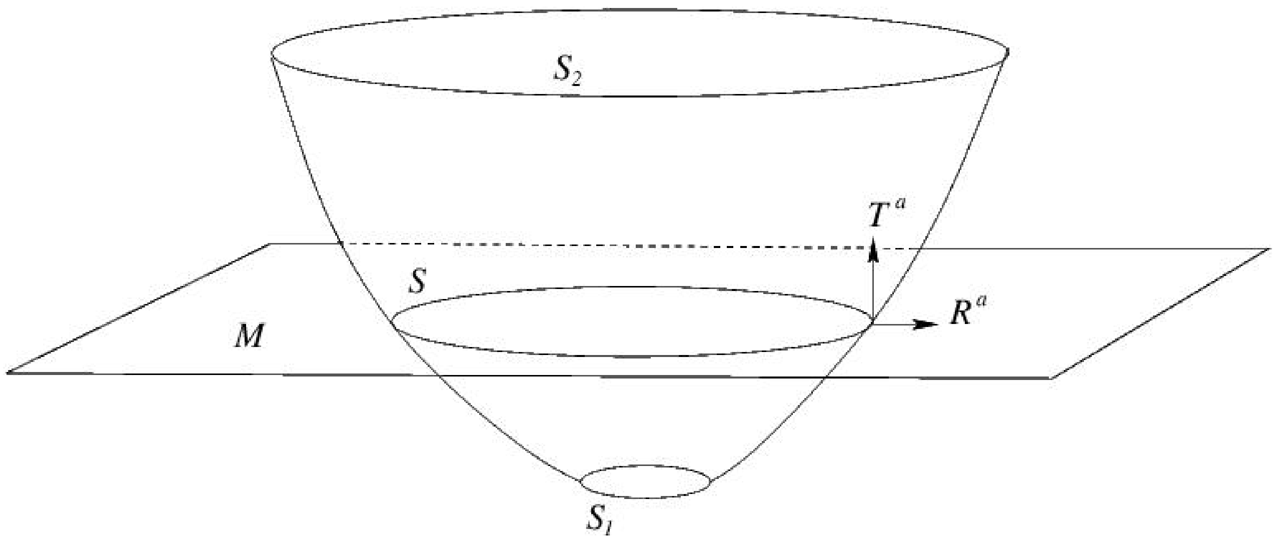
- While the mass M_S can easily be evaluated by:

$$M_S = \frac{1}{2GR_\Delta}\sqrt{R_\Delta^4 + 4G^2 J_\Delta^2} \tag{15}$$

where R_Δ is the areal radius of the horizon, $R_\Delta = (a_\Delta/4\pi)^{1/2}$

Applications in numerical relativity

- Numerical simulations, on the other hand, deal with the 3-metric $\bar{q}_{ab}$ and extrinsic curvature $\bar{K}_{ab}$ on (partial) Cauchy surfaces M.



The world tube of apparent horizons and a Cauchy surface M intersect in a 2-sphere S. T^a is the unit time-like normal to M and R^a is the unit space-like normal to S within M.

- Recasting the angular formula in terms of the Cauchy data $(\bar{q}_{ab}, \bar{K}_{ab})$ one gets:

$$J_\Delta = -\frac{1}{8\pi G}\oint_S \varphi^a R^b \bar{K}_{ab} d^2V \qquad (16)$$

where φ^a is a rotational Killing field for the intrinsic 2-metric on the cross-sections S.

Applications in numerical relativity

- This formula is particularly convenient numerically since it involves the integral of a single component of the extrinsic curvature.

- It is in fact applicable in both the isolated and dynamical regimes.

- It differs from the standard formula for the ADM angular momentum only in that the integral is over the apparent horizon instead of the sphere at infinity.

- Comparison with other methods:Standard procedures of calculating angular momentum were based on properties of the Kerr geometry.

- Motivation: a black hole created in a violent event radiates away all its higher multipole moments and as it settles down, its near horizon geometry can be approximated by that of the Kerr solution.

- Strategy: identify the geometry of S with that of a member of the Kerr family and read off the corresponding angular momentum and mass parameters. Note that this is still a non-trivial task since usually the coordinate system used does not bear any relation to any of the standard coordinate systems used to describe the Kerr solution.

- It is not suitable in the dynamical regime where the horizon may be distorted and not well-approximated by any Kerr horizon.

Applications in numerical relativity

- On the other hand the approach using the Isolated and Dynamical horizon spacetime notions has, from the 'practical' point of view, three main features:
  - The procedure does not presuppose that the horizon geometry is precisely that of the Kerr horizon.
  - It is coordinate independent.
  - It only requires data that is intrinsic to the apparent horizon.
- All of these features are not shared by none of the usual alternative approaches.

Conclusion

- The isolated and dynamical horizon frameworks provide a quasi-local and more physical paradigm for describing black holes both in the equilibrium and dynamical regimes.

- This new paradigm also provides a unified approach to a variety of problems involving black holes, ranging from entropy calculations in quantum gravity, to analytical issues related to numerical simulations.

- More importantly, these frameworks provide brand new laws in the dynamical regime.