Time functions in Numerical Relativity

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Abstract

We investigate in this paper the existence and uniqueness of time functions determined by maximal ($K = 0$) and constant-mean curvature ($K = K_0$) slicings. Some concepts necessary to study the causal structure of spacetimes are introduced. A new kind of singularity, *crushing singularity*, is defined and its properties reviewed. At last, Tolman-Bondi spacetimes were used as a laboratory for investigating under what conditions $K = 0$ and $K = K_0$ slicings will avoid the singularities.

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1 Motivation

3 + 1 formalism cast Einstein's equations as an initial value problem. In order to construct the solutions of this problem (to build the entire spacetime), a selection of a good spacetime coordinate system is needed to describe the dynamics of the slices. In particular, we are going to focus on the selection of the hypersurface s themselves, via the definition of some time function. Historically maximal slicing has been a popular choice. Mainly for the following two reasons: Simplifies somewhat the equations of motion and the constraints and it has been found that it "avoids" the singularity. In this paper I will investigate when this choice is possible for spacetimes in the presence of singularities.

2 Causal Structure

Causal approach is concerned with the detection by any observer that his spacetime is singular. In order to study the causal properties of the spacetimes generated by numerical relativity some definitions are required.

2.1 Domain of influence or chronological domain

The basic notion of the domain of influence, 'precede', reflects the physical question of whether or not event p can influence event q by means of a signal. It is then a relation between single points. For p and q any two points of M, we say that p chronologigally precedes q if there exists a future-directed timelike curve which begins at p and ends at q. Then $q \in I^+(p)$

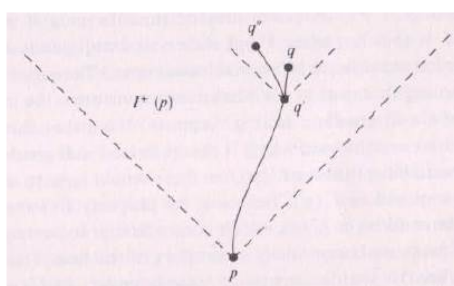

Figure 1: The chronological future $I^+(p)$ is the region a particle in p can affect.

2.2 Domain of dependence

A closely related question: given information in some region of spacetime when will it determine the physical situation in some other region? In order to determine what is going to happen on point p all the signals that could influence the physics at p has to be taken into account. The idea of the definition of future domain of dependence is this. Regard signals in relativity as travelling along non-spacelike curves. By demanding that all non-spacelike curves reaching p also meet S, one is ensuring that all signals which could influence physics at p are registred on S. Hence the physical situation in $D^+(S)$ will be completely determined by information on S.

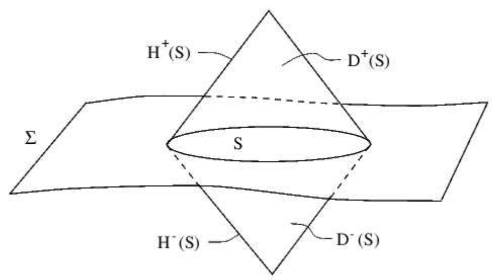

Figure 2: The future domain of dependence $D^+(S)$ is the region in which its physics is completely determined by the information on surface S.

2.3 Cauchy Horizons

Future (past) Cauchy horizon of S are defined as the boundary of the domain of dependence of S: $H^{\pm}(S) = \bar{D}^{\pm}(S) - I^{\mp}[D^{\pm}(S)]$. The the future Cauchy horizon, for instance, would be the closure of the domain of dependence of S minus the chronological past of this domain of dependence as can be seen on figure 2.

Let's move on to a classical example of Cauchy horizon. Consider Minkowski spacetime. The way it can be sliced is highly non-unique. In particular, it can be sliced by the hyperboloid Σ (fig.3). This hyperboloid has asymptotes $t = \pm x$. To determine the domain of dependence it is enough to find the points in spacetime in which every non-spacelike curve passing through them also intersects the hypersurface S. As can be seen on figure 3, it is possible to draw timelike curves, for example, that do not cross the hyperboloid. Then the region between the hyperboloid and its asymptotes is its the domain of dependence and its asymptotes its Cauchy horizons.

It is worth noting however that although the hyperboloid is a valid spacelike hypersurface to be used in the spacetime slicing, this foliation would never cover the entire Minkowski spacetime. Its domain of dependence does not cover the spacetime: $D(S) \neq M$. In this case the hyperboloid belongs to a class of surfaces called *partial Cauchy surface*. On the other hand, if $D(S) = M$, as it would be the case for $t =$ contant slicing, the surface S would be called *Cauchy surface* and M a *globally hyperbollic spacetime*.

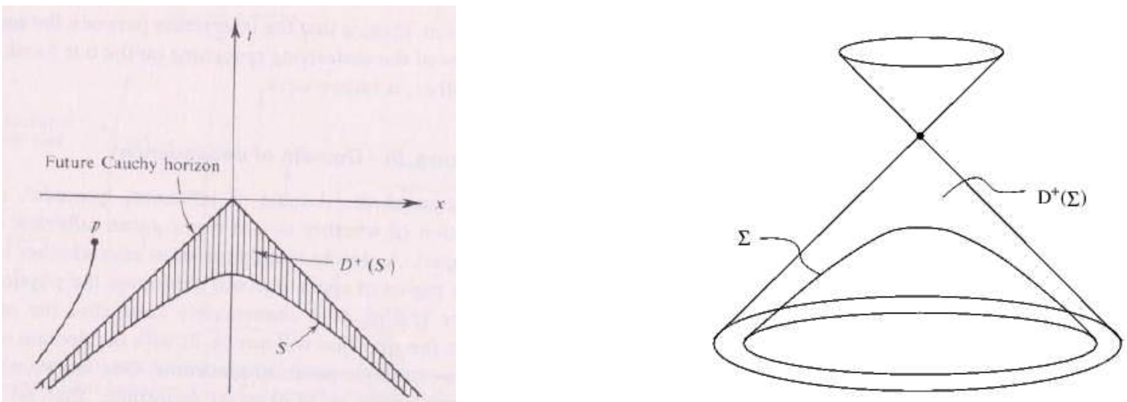

Figure 3: Hyperboloid slicing of Minkowski spacetime.

As seen above, the choice of the hypersurface is fundamental to determine the region of the spacetime will be covered by the foliation. Nonetheless, the way those hypersurfaces are evolved, i.e., the way time functions are chosen is also crucial. Let's then define time function and some correlated definitions.

2.4 Time functions

Consider a general spacetime (M, g_{ab}). A time function in a open region N of M is a real valued function with a future directed timelike gradient vector $-\nabla^a t$. A Cauchy time function, on the other hand, is a time function such that any inextendible non-spacelike curve intersects S exactly once. S is called then a Cauchy slice for N. The spacetime N is the *future development* $d_t^+(S)$ of the initial slice S, i.e., a solution of the Einstein equations which contains S as a Cauchy slice, therefore N is globally hyperbolic. Note that the numerical relativity prescription for constructing spacetimes generates only globally hyperbolic ones.

As pointed out before a bad choice of time function could lead to the situation of the future development not covering the entire spacetime neither the domain of dependence of S. Back to the hyperboloid example, the hypersurface is a bad choice since it doesn't cover the entire spacetime. However, a bad choice of time function could make this scenario even worse. For example the slices could asymmetrically pile out on the right region such that $d_t^+(S) \subset D^+(S)$. As for different specifications of the lapse function lead to different future developments, the question that arises is if there is a prescription to construct maximal developments ($d_{max}(S) \equiv D(S)$)?

3 Maximal and constant-mean-curvature slices

The answer for the last equation is no! There is no prescription to find the ideal time function that would cover the interior and exterior of a black hole for example. However one interesting class of time function, the Maximal ($K = 0$) and constant-mean-curvature slicings ($K = K_0$), has been used in the past with promising results. It simplifies the Einstein equations and have a nice property of avoindance of singularities for certain spacetimes. The remain part of this paper will be focusing on what is known about these slicings and testing their properties on known spacetimes.

The extrinsic curvature K is defined as:

$$K = \nabla_\mu[\nabla^\mu t/(-g_{\nu\lambda}\nabla^\nu t\nabla^\lambda t)^{1/2}] \tag{1}$$

In order to study numerically spacetimes using those time functions, some questions should be addressed: Can one find a single spacelike hypersurface S in the spacetime for which $K(S) = K_0$? This is the slice on which one wishes to pose initial data. If so, does there exist a family of such slices and is the future boundary of $d_t^+(S)$ by $K(S) = K_0 \geq 0$ time function t nonsingular? This is the evolution of the initial data. Does $d_t^+(S) = d_{max}^+(S)$? How much of the domain of dependence of S can be reached by t?

The first two questions poses the problem of uniqueness and existence of those slices. Motivated by numerical evidence, some theorems and conjectures were stablished. However those theorems do not encompass all classes of spacetimes. Their validity is limited to some cosmologies and assymptotically flat spacetimes. Another strategy adopted [1] in search for answers to those questions consists in characterizing the singularity structure of a class of spacetimes that the existence and uniqueness is known. Then the problem becomes to discover how broad is this class of "crushing singularities" among all the singularities that arises in the gravitational collapse. Next section "crushing singularity" will be defined and to gain some insight about this class of spacetimes the spherically symmetric dust spacetimes of Tolman and Bondi will be discussed.

4 Crushing singularities and avoidance theorems

Singularity in Numerical Relativity means future (or past) boundary of $D(S)$, excluding the infinities $\mathcal{I}$'s. Then part of $H^{\pm}(S)$ may be considered as part of the singularity. It's reasonable to define in this way since the 3-metric γ_{ij} becomes singular at $\dot{D}(S)$ Even if the singularity is inside the black hole, one still has the problem that various time functions t lead to different future boundaries on $d_t^+(S)$. This leads to different scenarios: The time slices of t can hit the singularity at a point or small neighbourhood in $S(t)$ They can uniformly wrap up around the singularity They cannot probe an open neighbourhood of the singularity ("avoidance of singularity") Then a reasonable question to ask is what would be the behaviour of Cauchy time functions near such singularities?

"Natural time functions" approaches such boundaries (singularities) uniformly for some model spacetimes such as in Friedman ($\tau =$ const., true spacetime singularity) and $r =$ const. in Schwartzschild (singul.) and Reissner-Nordstrom (Cauchy horizon). The strategy then is to define a class of spacetimes including this behavior for time functions. Then the Cauchy time function that hit the singularity uniformly will be called *future crushing function.* In a more formal way: let $c < f < 0$, if $lim K = \infty$ as $f \to 0^-$, then f is a *future crushing function.*

It can be proved that for a spatially compact, globally hyperbolic spacetimes that has future crushing singularities there exists a Cauchy constant-mean-curvature time function t such that its slices wrap up around the singularities uniformly. This result was first observed for the Schwartzschild black hole and easily stablished later on for the other two well known stationary spacetimes in the presence of singularities: Kerr and Reissner-Nordström spacetimes.

5 Tolman-Bondi (TB) spacetimes [2, 3]

Initially proposed as a set of inhomogeneous, spherical symmetric solutions for dust. Actually it groups under the same metric well known spherical symmetric solutions of Einstein's equations either for dust or vacuum mather model. For instance: Schwartzschild black hole, the homogeneous Friedman universes, the Oppenheimer-Snyder star collapse, etc. The great advantage of discussing TB spacetimes is that the exact form of the metric is know for the entire spacetime. For this reason, it becomes a good laboratory for testing $K = 0$ and $K = K_0$ slincings in the presence of all sorts of singularities. We are going to focus on the marginally bound collapse.

5.1 Review of the metric

The general spherical symmetric metric is given by (in comoving coordinates):

$$ds^2 = -dt^2 + X^2(t,r)dr^2 + Y^2(r,t)d\Omega^2 \tag{2}$$

Taking into account a spherically symmetric dust is being modeled, i.e. using the following stress-energy tensor in comoving coordinates:

$$T^{tt} = \rho \tag{3}$$

The Einstein equations become:

$$X(r,t) = \frac{1}{W(r)}\frac{\partial Y(r,t)}{\partial r} \tag{4}$$

$$(\frac{\partial Y}{\partial t})^2 = W^2(r) - 1 + \frac{2}{Y(r,t)}\int_0^r \frac{dM(r')}{dr'}W(r')dr' \tag{5}$$

Where $M(r)$ is the total proper mass of the matter within a shell labeled by the comoving coordinate $r =$ const. and is given by (00 component of the Einstein field equations):

$$M'(r) = 4\pi\rho(r,t)X(r,t)Y^2(r,t) \tag{6}$$

and it allows us to interpret $4\pi Y^2(r,t)$ as the proper area of the shell. In order to have some intuition about the interpretation of the constant of integration $W(r)$, let's compare to the Newtonian case: Assume $W^2(r) = 1 + 2E(r)$ and $E(r)$ small and plug back in the second Einstein field equa tions:

$$\frac{1}{2}(\frac{\partial Y}{\partial t})^2 - \frac{M(r)}{Y} = E(r) \tag{7}$$

that is the familiar Newtonian energy equation. $W(r)$ can then be interpreted as the ratio between the binding energy $E(r)$ and the mass inside the shell $M(r)$. Only the marginally bound case will considered below ($W(r) = 1$, $0 \leq r < \infty$).

Integrating the $Y(r,t)$ equation and expressing in terms of t we get:

$$[t - t_0(r)]^2 = \frac{2}{9}\frac{Y^3(r,t)}{M(r)} \tag{8}$$

Since the area of the mass shell of constant r goes to zero when $Y(r,t) \to 0$ then $t_0(r)$ can be interpreted then as the time the mass shell takes to hit the singularity. As r just label the shells, we still have the coordinate freedom to relabel them for any other function of r. In particular, we can use this freedom to fix t_0 and M. If we choose $t_0'(r) = t_0(r) = 0$ and $M(r) = r^3$ gives the marginally bound (k=0) Friedman solution:

$$ds^2 = -dt^2 + (9t^2/2)^{2/3}(dr^2 + r^2 d\Omega^2) \tag{9}$$

On the other hand, for $M'(r) = 0$ and $t_0 = r$ we have the Eddington-Finkelstein patch of the extended Schwarztshild, written in Lemaitre coordinates:

$$ds^2 = -dt^2 + [4M/3(r-t)^{-1}]^{2/3}dr^2 + [9M/2(r-t)^2]^{2/3}d\Omega^2 \tag{10}$$

When glued together at $r = 1$, these spacetimes give the Oppenheimer-Snyder solutionof a homogeneous collapsing bound dust cloud. Before showing some numerical results let me summarize the causal structure of the marginally bound, asymptotically flat TB spacetimes in the form of Penrose-Carter diagrams

5.2 Causal structure

The first diagram, of three possibilities for inhomogeneous marginally bound dust-sphere collapse, represent a generalization of the Oppenheimer-Snider collapse:

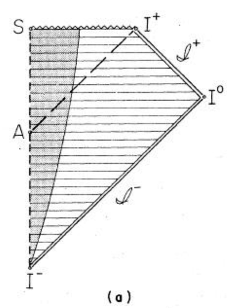

Figure 4: Oppenheimer-Snyder collapse causal structure.

In the dark grey region $T_{\mu\nu} \neq 0$. This is the interior region of the cloud of dust. Since the space time is asymptotically flat, we have the usual $I^{\pm}$ denoting the future and past infinities, $\mathcal{I}^+$ and $\mathcal{I}^-$ representing the future and the past null infinities, respectively and I^0 being the spacial infinity. The horizontally striped region represents the hyperbolic region in which numerical relativity has access. Note that for this causal structure the foliation cover all the spacetime, inside and outside the blackhole. The dash-line $A - I^+$ is the event horizon. For this causal structure the singularity "grows faster than light speed", forming a spacelike singularity everywhere, inclusive at $r = 0$. That means that no light ray can escape the singularity after it is formed. Mathematically, this qualitative behaviour is described by lim $t_0/M = 0$ as $r \to 0^+$.

If lim $t_0/M = \infty$ as $r \to 0^+$, then there is a piece of past-null singularity at the origin. This time the singularity "grows slower than speed of light", allowing light rays to scape from the singularity. The two possibilities are shown below.

Figure 5: (b)local naked singularity; (c)global naked singulaity. Note that in (b) we have the Cauchy horizon $(S - T)$ inside the event horizon $(A - I^+)$ while in (c) the event horizon isn't even defined.

In (b) rays coming from past null singularity are trapped inside the black hole and no naked singularity is visible from infinity. However, a break of predictability inside the black hole is observed. In this case we have a local naked singularity. On the other hand in (c), as the singularity grows slower, some of the rays scape from $r = 0$ singularity and reach the null infinity, characterizing then what is called global naked singularity. These two kinds of naked singularities are called collectivelly *shell-focusing singularity*. All these three cases can occur in the lim $t_0/M =$ finite constant.

In the next section some numerical results are shown using maximal slicing time function for a marginally bound, asymptotically flat TB spacetime. We focus on contrasting two particular solutions of TB spacetimes in order to show the failure of maximal slicing in covering the region of the spacetime outside the black hole.

5.3 Some numerical results

First consider the maximal slicing of $M = 1$ Oppenheimer-Snyder collapse of a marginally bound dust cloud:

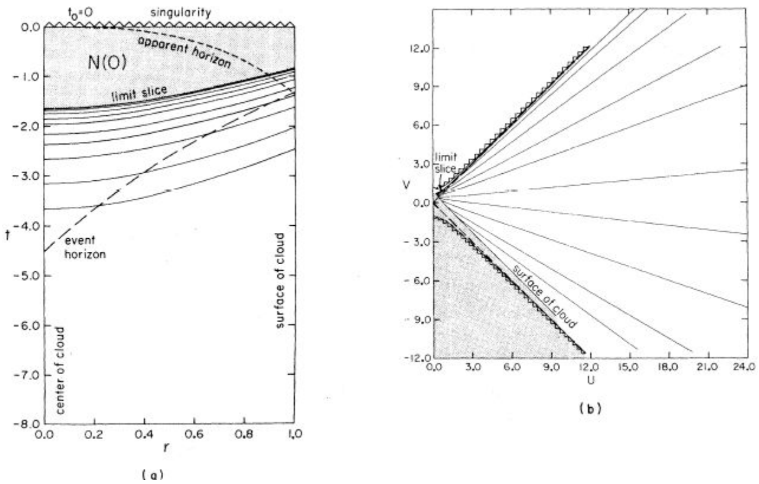

Figure 6: Maximal slices avoid the singularity and cover all the external region of the black hole in the Oppenheimer-Snyder collapse.

It is shown in (a) the interior of the dust cloud in TB coordinates. Note that as time goes on the slices converge to a limiting slice and avoid the singularity. As the slices leave the interior of the dust they emerge (b) from the interior into a Eddington-Finkelstein patch of $M = 1$ Schwartzschild spacetime represented in Kruskal coordinates. These slices avoid cover the entire region of the spacetime outside the black hole.

On the other hand if the collapse is too inhomogeneous there will be no guarantee that maximal slicing would work. In this particular example, the TB spacetime has the same 'nice' causal structure of the Oppenheimer-Snyder spacetime , but there is no crushing singularity. As we can see from the fig. 7, the slices hit the singularity and fails to cover all the spacetime outside the black hole. Therefore is not enough to state that the spacetime is globally hyper bollic with a totally spacelike singularity to guarantee that $K = 0$ will work. A stronger condition is required.

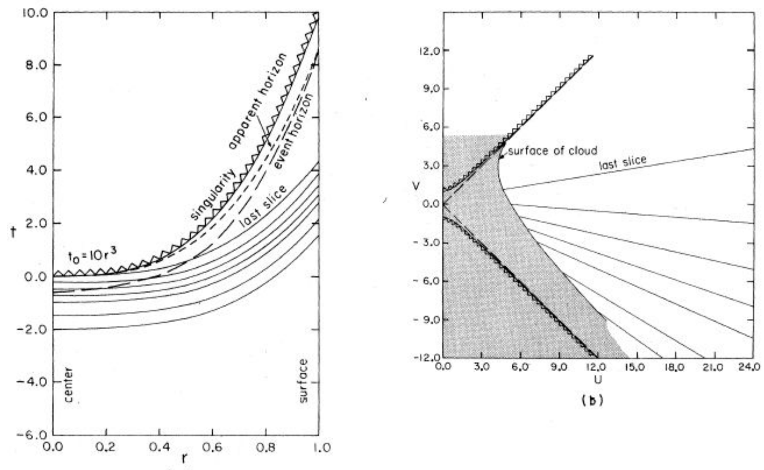

Figure 7: Maximal slices hit the singularity and doesn't cover the region outside the black hole in a too inhomogeneous collapse.

6 Conclusion

According to the avoidance theorems setting a globally hyperbollic, asymptotically flat spacetime with a spacelike singularity everywhere would be enough to guarantee that maximal slicing would work: avoid the singularity and cover all the region of the spacetime exterior to the black hole. However as the investigation of Tolman-Bondi spacetimes has shown further assumptions have to be taken, in this case the hypothesis of crushing singularities.

Up to date no general theorem about avoidance of singularity by maximal slicing and constant curvature has been proved. A possible theorem would necessarily assume stronger conditions than just the usual energy conditions. As nobody knows what conditions it would be, the cases in which those slices work will be determined exclusively by numerical search, with no theoretical guarantee.

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