Time functions in Numerical Relativity

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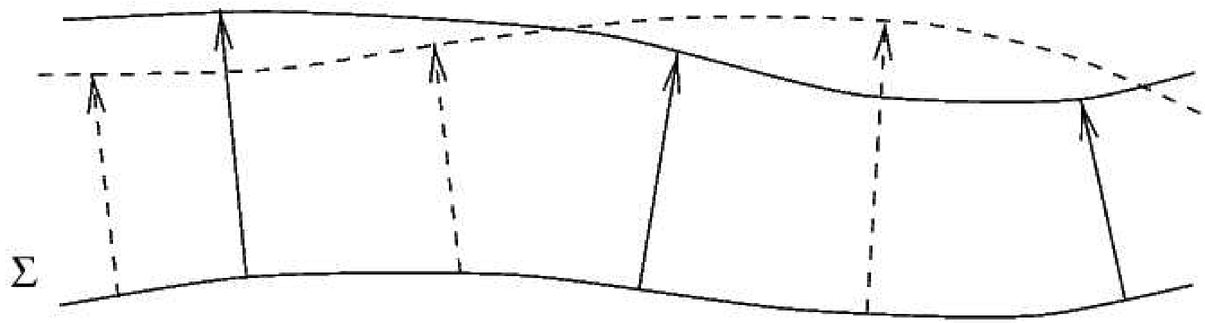
Motivation



- $3 + 1$ formalism cast Einstein's equations as an initial value problem.

- In order to construct the solutions of the problem (to build the entire spacetime), a selection of a good spacetime coordinate system is needed to describe the dynamics of the slices.

- In particular, we are going to focus on the selection of the hypersurfaces themselves, via the definition of some time function.

- Historically maximal slicing has been a popular choice. Mainly for the following two reasons:

  - Simplifies somewhat the equations of motion and the constraints.
  - It has been found that it "avoids" the singularity.

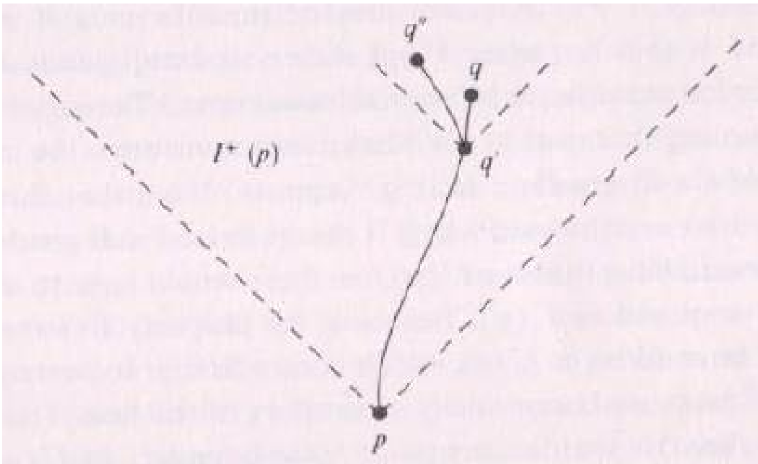
- In this talk I will investigate when this choice is possible for spacetimes in the presence of singularities.

Causal Structure

- Causal approach is concerned with the detection by any observer that his spacetime is singular. In order to study the causal properties of the spacetimes generated by numerical relativity some definitions are required.

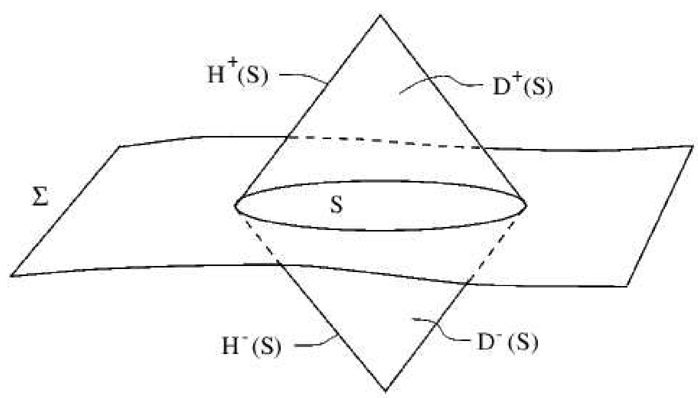
- Domain of influence or chronological domain

  - The basic notion of the domain of influence, 'precede', reflects the physical question of whether or not event p can influence event q by means of a signal. It is then a relation between single points.
  - For p and q any two points of M, we say that p chronologigally precedes q if there exists a future-directed timelike curve which begins at p and ends at q. Then $q \in I^+(p)$



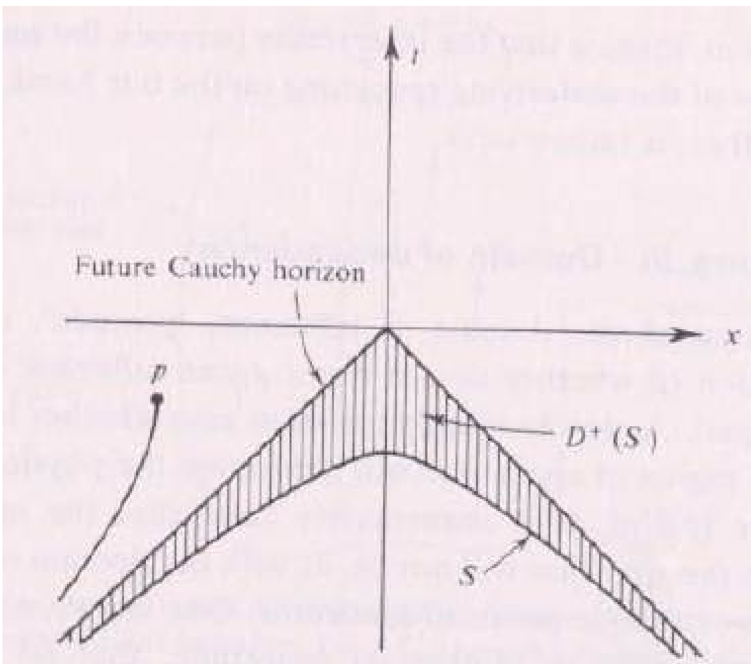
Causal Structure

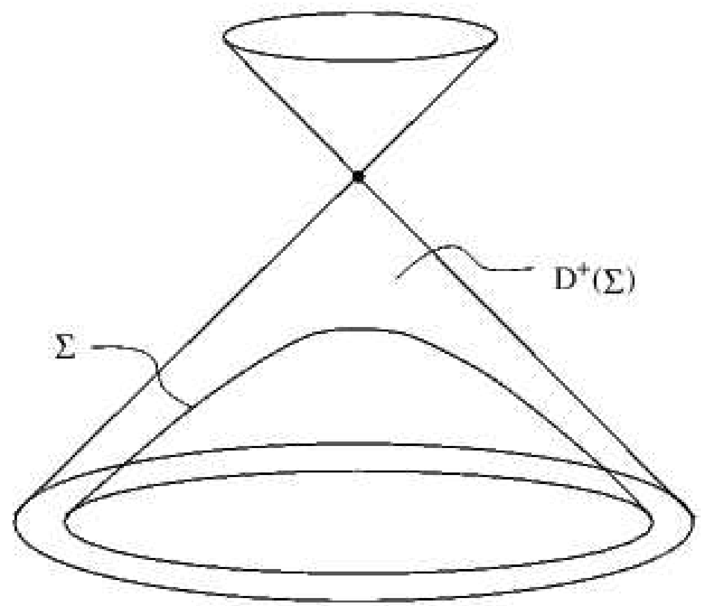
- Domain of dependence
  - A closely related question: given information in some region of spacetime when will it determine the physical situation in some other region?
  - In order to determine what is going to happen on point p all the signals that could influence the physics at p has to be taken into account.
  - The idea of the definition of future domain of dependence is this. Regard signals in relativity as travelling along non-spacelike curves. By demanding that all non-spacelike curves reaching p also meet S, one is ensuring that all signals which could influence physics at p are registred on S. Hence the physical situation in $D^+(S)$ will be completely determined by information on S.



Causal Structure

- Cauchy Horizons
  - Future (past) Cauchy horizon of S are defined as the boundary of the domain of dependence of S: $H^{\pm}(S) = \bar{D}^{\pm}(S) - I^{\mp}[D^{\pm}(S)]$
  - Consider Minkowski spacetime sliced by the hyperboloid Σ. It is a valid spacelike hypersurface however would not cover the entire spacetime. It would pile out before the Cauchy horizon.





- If $D(S) = M$, then S is called *Cauchy surface* and M a *globally hyperbollic spacetime*. Otherwise, if $D(S) \neq M$, S is a *partial Cauchy surface*

Causal Structure

- Time functions

- Consider a general spacetime (M, g_{ab}). A time function in a open region N of M is a real valued function with a future directed timelike gradient vector $-\nabla^a t$.

- Cauchy time function: time function such that any inextendible non-spacelike curve intersects S exactly once. S is a Cauchy slice for N.

- The spacetime N is the *future development* $d_t^+(S)$ of the initial slice S, i.e., a solution of the Einstein equations which contains S as a Cauchy slice, therefore N is globally hyperbolic.

- Back to the hyperboloid example, a bad time function could be chosen such that $d_t^+(S) \subset D^+(S)$

- As for different specifications of the lapse function lead to different future developments, the question that arises is if there is a prescription to construct maximal developments $(d_{max}(S) \equiv D(S))$?

Maximal and constant-mean-curvature slices

- The answer for the last equation is no! There is no prescription to find the ideal time function that would cover the interior and exterior of a black hole for example.

- One interesting class of time function is the Maximal $(K = 0)$ and constant-mean-curvature slicings $(K = K_0)$. Where the extrinsic curvature K is defined by:

$$K = \nabla_\mu[\nabla^\mu t/(-g_{\nu\lambda}\nabla^\nu t\nabla^\lambda t)^{1/2}] \quad (1)$$

- In order to study numerically spacetimes using those time functions, some questions should be addressed:

  - Can one find a single spacelike hypersurface S in the spacetime for which $K(S) = K_0$? This is the slice on which one wishes to pose initial data.
  - If so, does there exist a family of such slices and is the future boundary of $d_t^+(S)$ by $K(S) = K_0 \geq 0$ time function t nonsingular? This is the evolution of the initial data.
  - Does $d_t^+(S) = d_{max}^+(S)$?
  - How much of the domain of dependence of S can be reached by t?

Maximal and constant-mean-curvature slices

- The first two questions poses the problem of uniqueness and existence of those slices.

- Motivated by numerical evidence, some theorems and conjectures were stablished.

- However those theorems do not encompass all classes of spacetimes. Their validity is limited to some cosmologies and assymptotically flat spacetimes.

- Another strategy in search for answers to those questions consists in characterizing the singularity structure of a class of spacetimes that the existence and uniqueness is known.

- Then the problem becomes to discover how broad is this class of "crushing singularities" among all the singularities that arises in the gravitational collapse.

- Next section "crushing singularity" will be defined and to gain some insight about this class of spacetimes the spherically symmetric dust spacetimes of Tolman and Bondi will be discussed.

Crushing singularities and avoidance theorems

- Singularity in Numerical Relativity means future (or past) boundary of $D(S)$, excluding the infinities $\mathcal{I}$'s. Then part of $H^{\pm}(S)$ may be considered as part of the singularity. It's reasonable to define in this way since the 3-metric γ_{ij} becomes singular at $\dot{D}(S)$

- Even if the singularity is inside the black hole, one still has the problem that various time functions t lead to different future boundaries on $d_t^+(S)$. This leads to different scenarios:

  - The time slices of t can hit the singularity at a point or small neighbourhood in $S(t)$
  - They can uniformly wrap up around the singularity
  - They cannot probe an open neighbourhood of the singularity ("avoidance of singularity")

- What's the behaviour of Cauchy time functions near such singularities?

Crushing singularities and avoidance theorems

- "Natural time functions" approaches such boundaries (singularities) uniformly for some model spacetimes such as in Friedman ($\tau =$ const., true spacetime singularity) and $r =$ const. in Schwartzschild (singul.) and Reissner-Nordstrom (Cauchy horizon)

- This feature can be generalized: *"crushing singularity"*

- The Cauchy time function that hit the singularity uniformly is called *future crushing function*: Let $c < f < 0$ then $limK = \infty$ as $f \to 0^-$.

- It can be proved that for a spatially compact, globally hyperbolic spacetimes that has future crushing singularities there exists a Cauchy constant-mean-curvature time function t such that its slices wrap up around the singularities uniformly.

- This result was first observed for the Schwartzschild black hole.

Tolman-Bondi spacetimes

- Initially proposed as a set of inhomogeneous, spherical symmetric solutions for dust.
- Actually it group under the same metric well known spherical symmetric solutions of Einstein's equations either for dust or vacuum mather model.
- For instance: Schwartzschild black hole, the homogeneous Friedman universes, the Oppenheimer-Snyder star collapse, etc.
- The great advantage of discussing TB spacetimes is that the exact form of the metric is know for the entire spacetime.
- For this reason, it becomes a good laboratory for testing $K = 0$ and $K = K_0$ slincings in the presence of all sorts of singularities.
- We are going to focus on the marginally bound collapse.

Tolman-Bondi spacetimes

- Review of the metric

  - The general spherical symmetric metric is given by (in comoving coordinates):
$$ds^2 = -dt^2 + X^2(t,r)dr^2 + Y^2(r,t)d\Omega^2 \tag{2}$$
  - Taking into account a spherically symmetric dust is being modeled, i.e. using the following stress-energy tensor in comoving coordinates:
$$T^{tt} = \rho \tag{3}$$
  - The Einstein equations become:
$$X(r,t) = \frac{1}{W(r)}\frac{\partial Y(r,t)}{\partial r} \tag{4}$$
$$(\frac{\partial Y}{\partial t})^2 = W^2(r) - 1 + \frac{2}{Y(r,t)}\int_0^r \frac{dM(r')}{dr'}W(r')dr' \tag{5}$$

Tolman-Bondi spacetimes

- Where $M(r)$ is the total proper mass of the matter within a shell labeled by the comoving coordinate $r =$ const. and is given by (00 component of the Einstein field equations):

$$M'(r) = 4\pi\rho(r,t)X(r,t)Y^2(r,t) \quad (6)$$

- and it allows us to interpret $4\pi Y^2(r,t)$ as the proper area of the shell.

- In order to have some intuition about the interpretation of the constant of integration $W(r)$, let's compare to the Newtonian case: Assume $W^2(r) = 1 + 2E(r)$ and $E(r)$ small and plug back in the second Einstein field equa tions:

$$\frac{1}{2}(\frac{\partial Y}{\partial t})^2 - \frac{M(r)}{Y} = E(r) \quad (7)$$

- that is the familiar Newtonian energy equation. $W(r)$ can then be interpreted as the ratio between the binding energy $E(r)$ and the mass inside the shell $M(r)$. Only the marginally bound case will considered below ($W(r) = 1$, $0 \leq r < \infty$).

Tolman-Bondi spacetimes

- Integrating the $Y(r,t)$ equation and expressing in terms of t we get:

$$[t - t_0(r)]^2 = \frac{2}{9}\frac{Y^3(r,t)}{M(r)} \tag{8}$$

- Since the area of the mass shell of constant r goes to zero when $Y(r,t) \rightarrow 0$ then $t_0(r)$ can be interpreted then as the time the mass shell takes to hit the singularity.

- As r just label the shells, we still have the coordinate freedom to relabel them for any other function of r. In particular, we can use this freedom to fix t_0 and M. If we choose $t_0'(r) = t_0(r) = 0$ and $M(r) = r^3$ gives the marginally bound (k=0) Friedman solution:

$$ds^2 = -dt^2 + (9t^2/2)^{2/3}(dr^2 + r^2 d\Omega^2) \tag{9}$$

Tolman-Bondi spacetimes

- On the other hand, for $M'(r) = 0$ and $t_0 = r$ we have the Eddington-Finkel stein patch of the extended Schwartzshild, written in Lemaitre coordinates:

$$ds^2 = -dt^2 + [4M/3(r-t)^{-1}]^{2/3} dr^2 + [9M/2(r-t)^2]^{2/3} d\Omega^2 \qquad (10)$$

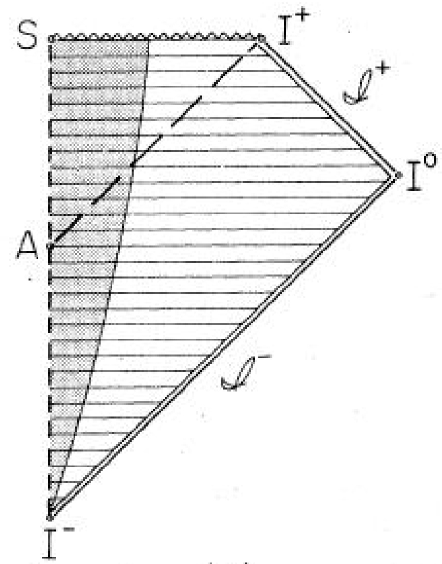
- When glue together at $r = 1$, we have the Oppenheimer-Snyder solution of a homogeneous collapsing bound dust cloud.

- Before showing some numerical results let me summarize the causal structure of the marginally bound, asymptotically flat TB spacetimes in the form of Penrose diagrams

Tolman-Bondi spacetimes

- Causal structure

- The first diagram, of three possibilities for inhomogeneous marginally bound dust-sphere collapse, represent a generalization of the Oppenheimer-Snider collapse:



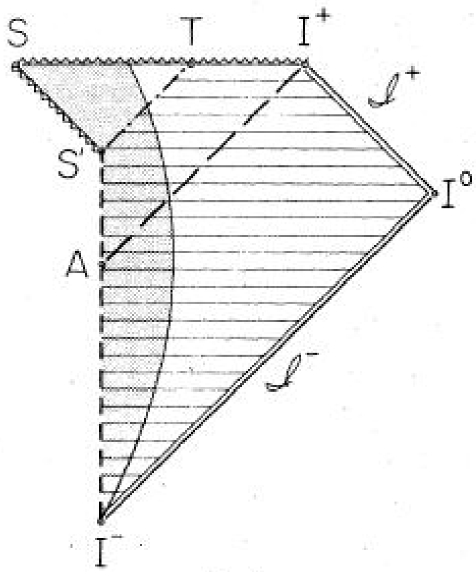
(a)

- In the dark grey region $T_{\mu\nu} \neq 0$. The horizontally striped region represents the hyperbolic region in which numerical relativity has access.

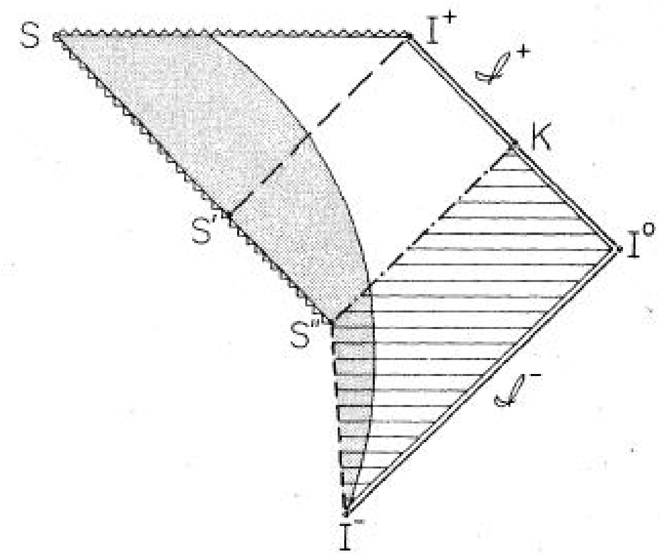
- Here the singularity "grows faster than light speed", forming a spacelike singularity at $r = 0$. It is characterized by lim $t_0/M = 0$ as $r \to 0^+$.

Tolman-Bondi spacetimes

- If lim $t_0/M = \infty$ as $r \to 0^+$, then there will be a piece of past-null singularity at the origin. This time the singularity "grows slower than speed of light", allowing light rays to scape to infinity.



(b)



(c)

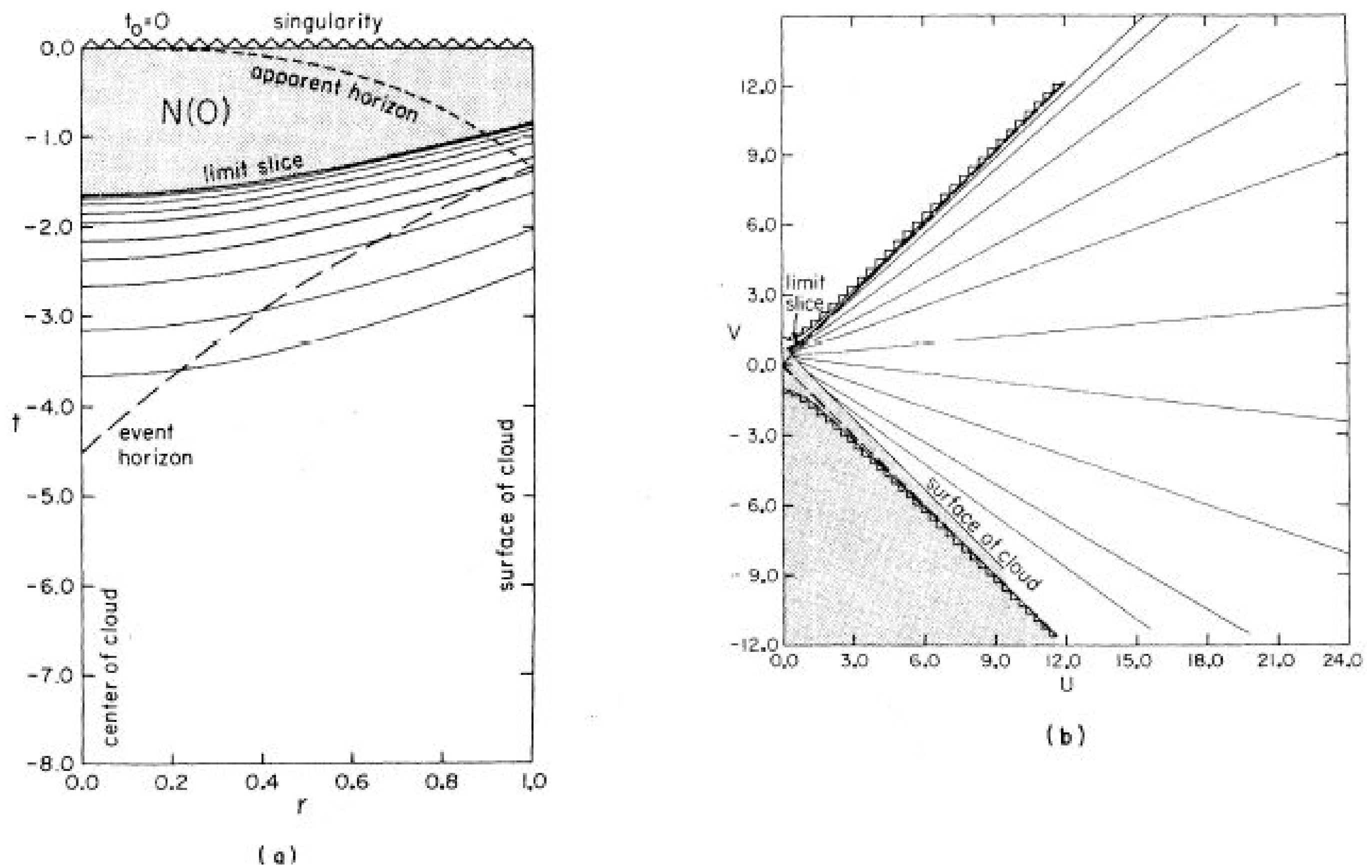
- In (b) those rays are trapped inside the black hole and no naked singularity is observed despite a break of predictability inside the black hole. In this case we have a local naked singularity.

- In(c) some of the rays reach the null infinity, then we have a global naked singularity. Those singularities are called shell-focusing singularity. All these three cases can occur in the lim t_0/M = finite constant.

Tolman-Bondi spacetimes

- Some numerical results

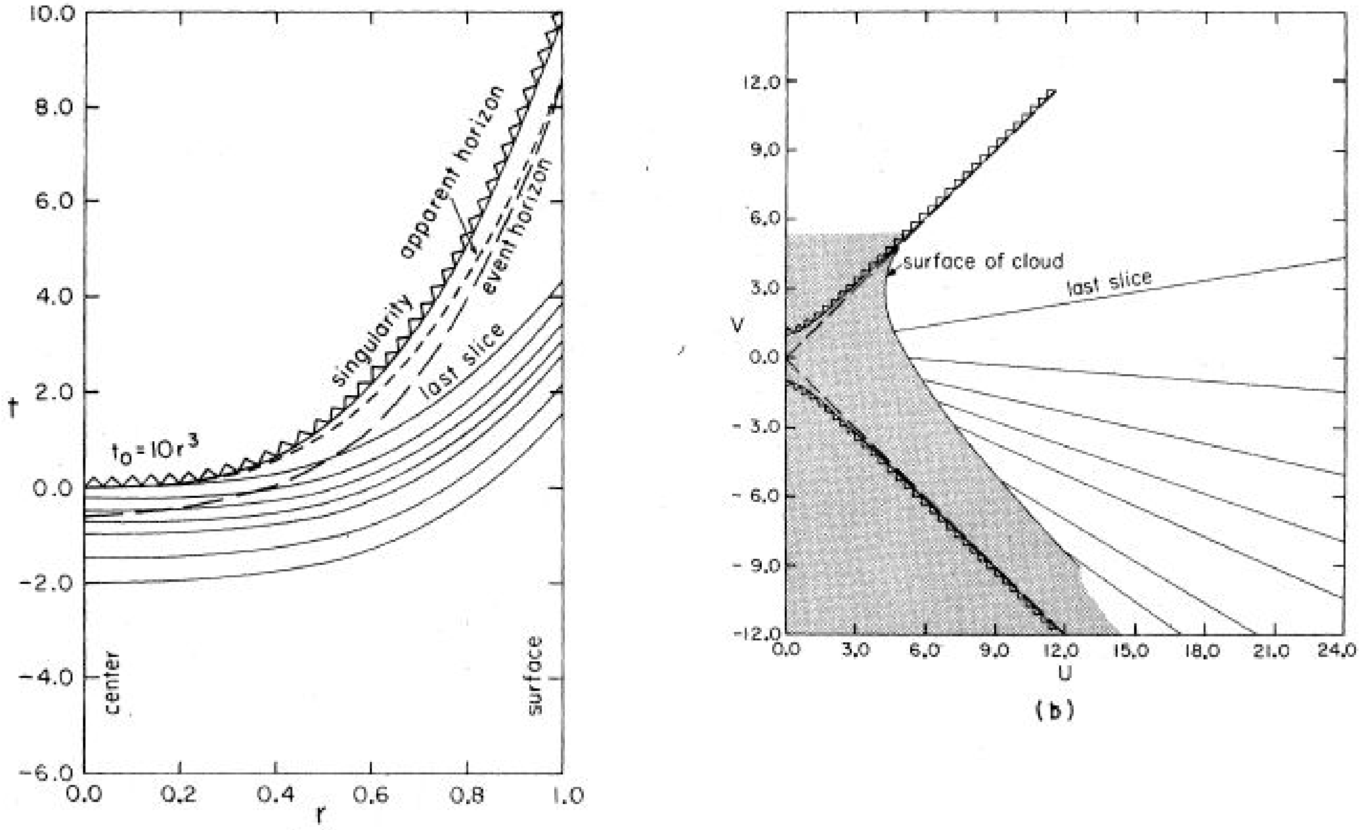
- Maximal slicing of Oppenheimer-Snyder collapse:



- It is shown in (a) the interior of the dust cloud in TB coordinates and in (b) as the slices emerge from the interior the Schwartzshild spacetime in Kruskal coordinates. Note that the slices avoid the singularity.

Tolman-Bondi spacetimes

- Maximal slicing failure for a too inhomogeneous collapse:



- Note that the causal structure is the same as Oppenheimer-Snyder case, however the slices hit the singularity and fails to cover all the spacetime of interest. Therefore is not enough to state that the spacetime is globally hyper bollic with a spacelike singularity everywhere to guarantee that $K = 0$ will work. A stronger condition is required.

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