Computational methods in general relativity - the theory

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I. CONVENTIONS AND UNITS

This article adopts many of the conventions and notations of Misner, Thorne and Wheeler (1973)—hereafter denoted MTW—including: metric signature $(- + + +)$; definitions of Christoffel symbols and curvature tensors (up to index permutations permitted by standard symmetries of the tensors in a coordinate basis); the use of Greek indices $\alpha, \beta, \gamma, \ldots$, ranging over the spacetime coordinate values $(0, 1, 2, 3) \to (t, x^1, x^2, x^3)$, to denote the components of *spacetime tensors* such as $g_{\mu\nu}$; the similar use of Latin indices $i, j, k, \ldots$, ranging over the *spatial* coordinate values $(1, 2, 3) \to (x^1, x^2, x^3)$, for *spatial* tensors such as γ_{ij}; the use of the Einstein summation convention for both types of indices; the use of standard Kronecker delta symbols (tensors), $\delta^\mu{}_\nu$ and $\delta^i{}_j$; the choice of geometric units, $G = c = 1$; and, finally, the normalization of the matter fields implicit in the choice of the constant 8π in (1).

The majority of the equations that appear in this article are tensor equations, or specific components of tensor equations, written in traditional index (*not* abstract index) form. Thus, these equations are generally valid in *any* coordinate system, (t, x^i), but, of course *do* require the introduction of a coordinate basis and its dual. This approach is also largely a matter of convention, since all of what follows can be derived in a variety of fashions, some of them purely geometrical, and there are also approaches to numerical relativity based, for example, on frames rather than coordinate bases.

This article *departs* from MTW in its use of α, β^i and γ_{ij} to denote the lapse, shift and spatial metric respectively, rather than MTW's N, N^i and ${}^{(3)}g_{ij}$.

Finally, the operations of partial differentiation with respect to coordinates x^μ, t and x^i are denoted ∂_μ, ∂_t and ∂_i, respectively.

II. INTRODUCTION

The numerical analysis of general relativity, or *numerical relativity*, is concerned with the use of computational methods to derive approximate solutions to the Einstein field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \,. \tag{1}$$

Here, $G_{\mu\nu}$ is the Einstein tensor—that contracted piece of the Riemmann curvature tensor that has vanishing divergence—and $T_{\mu\nu}$ is the stress tensor of the matter content of the spacetime. $T_{\mu\nu}$ likewise has vanishing divergence, an expression of the principle of *local* conservation of stress-energy that general relativity embodies.

The elegant tensor formulation (1) belies the fact that, ultimately, the field equations are generically a complicated and nonlinear set of partial differential equations for the components of the spacetime metric tensor, $g_{\mu\nu}(x^\alpha)$, in some coordinate system x^α. Moreover, implicit in a numerical solution of (1) is the numerical solution of the equations of motion for any *matter fields* that couple to the gravitational field—that is, that contribute to $T_{\mu\nu}$. The reader is reminded that it is a hallmark of general relativity that, in principle, *all* matter fields—including *massless* ones such as the electromagnetic field—contribute to $T_{\mu\nu}$.

Now, in the $3 + 1$ approach to general relativity that is described below, the task of solving the field equations (1) is formulated as an *initial value* or *Cauchy* problem. Specifically the spacetime metric, $g_{\mu\nu}(x^\alpha) = g_{\mu\nu}(t, x^k)$, which encodes *all* geometric information concerning the spacetime, $\mathcal{M}$, is viewed as the time history, or dynamical evolution, of the spatial metric, $\gamma_{ij}(0, x^k)$, of an initial spacelike hypersurface, $\Sigma(0)$. In any practical calculation, the degree to which the matter fields "backreact" on the gravitational field, that is contribute to $T_{\mu\nu}$ substantially enough to cause perturbations in $g_{\mu\nu}$ at or above the desired accuracy threshold, will thus depend on the specifics of the initial configuration.

In astrophysics, there are relatively few well identified environments in which it is generally thought to be crucial to the faithful emulation of the physics that the matter fields be fully coupled to the gravitational field. However, both observationally and theoretically, the existence of *gravitationally compact* objects is quite clear. Gravitationally compact means that a star with mass, M, has a radius, R, comparable to its Schwarzschild radius, R_M, which is defined by

$$R_M = \frac{2G}{c^2} M \approx 10^{-27}\,\mathrm{kg\,m^{-1}}\,. \tag{2}$$

Here, and only here, G and c—Newton's gravitational constant and the speed of light, respectively—have been explicitly reintroduced. The fact that R_M/R is about 10^{-6} and 10^{-9} at the surfaces of the sun and earth, respectively, is a reminder of just how weak gravity *is* in the locality of Earth. However, as befits anything of Einsteinian nature, the *weakness* of gravity is relative, so that at the surface of a neutron star, one would find

$$\frac{R_M}{R} \sim 0.4\,, \tag{3}$$

while for black holes, one has

$$\frac{R_M}{R} = 1\,. \qquad (4)$$

In such circumstances, gravity is anything but weak! Furthermore, in situations where the matter-energy distribution has a highly time dependent quadrupole moment—such as occurs naturally with a compact-binary system (i.e. a gravitationally bound two-body system, in which each of the bodies is either a black hole or a neutron star)—the *dynamics* of the gravitational field, including, crucially, the dynamics of the *radiative components* of the gravitational field, can be expected to dominate the dynamics of the overall system, matter included. For scenarios such as these, it should come as no surprise that the solution of the combined gravito-hydrodynamical system begs for numerical analysis.

In addition, both from the physical and mathematical perspectives, it is also natural to study the strong, field dynamic regimes ($R \to R_M$ and/or $v \to c$ where v is the typical speed characterizing internal bulk motion of the matter) of general relativity within the context of a variety of matter models. Typical processes addressed by these theoretical studies include the process of black hole formation, end-of-life-events for various types of model stars, and, again, the interaction, including collisions, of gravitationally compact objects. Note that it is another hallmark of general relativity that highly dynamical spacetimes need not contain *any* matter; indeed, the interaction of two black holes—the natural analog of the Kepler problem in relativity—is a *vacuum* problem; that is, it is described by a solution of (1) with $T_{\mu\nu} = 0$.

Motivated in significant part by the large scale efforts currently underway to directly detect gravitational radiation (gravitational waves), much of the contemporary work in numerical relativity is focused on precisely the problem of the late phases of compact-binary inspiral and merger. Such binaries are expected to be the most likely candidates for early detection by existing instruments such as TAMA, GEO, VIRGO, LIGO, and, more likely, by planned detectors including LIGO II and LISA (see for example Hough and Rowan (2000)). Detailed and accurate predictions of expected waveforms from these events—using the techniques of numerical relativity—has the potential to substantially hasten the discovery process, on the basis of the general principle that if one knows what signal to look for, it is much easier to extract that signal from the experimental noise.

The computational task facing numerical relativists who study problems such as binary inspiral is formidable. In particular such problems are intrinsically "3D", to use the CFD (computational fluid dynamics) nomenclature in which time dependence is always assumed. That is, the PDEs (partial differential equations) that must be solved govern functions, $F(t, x^k)$, that depend on all three spatial coordinates, x^k, as well as on time, t. Unfortunately, even a cursory description of 3D work in numerical relativity as it stands at this time is far beyond the scope of this article.

What follows, then, is an outline of a traditional approach to numerical relativity that underpins many of the calculations from the early years of the field (1970's and 1980's), most of which were carried out with simplifying restrictions to either spherical symmetry or axisymmetry. The mathematical development, which will hereafter be called the 3+1 approach to general relativity, has the advantage of using tensors and an associated tensor calculus that are reasonably intuitive for the physicist. This "standard" 3 + 1 approach is also *sufficient* in many instances (particularly those with symmetry) in the sense that it leads to well-posed sets of PDEs that *can* be discretized and then solved computationally in a convergent (stable) fashion. In addition, a thorough understanding of the 3 + 1 approach will be of significant help to the reader wishing to study any of the current literature in numerical relativity, including the 3D work.

However, *the reader is strongly cautioned* that the blind application of *any* of the equations that follow, especially in a 3D context, may well lead to *ill posed systems*, numerical analysis of which is useless. Anyone specifically interested in using the methods of numerical relativity to generate discrete, approximate solutions to (1), particularly in the generic 3D case, is thus urged to first consult one of the comprehensive reviews of numerical relativity that continue to appear at fairly regular intervals (see for example, Lehner (2001), or Baumgarte and Shapiro (2003)). Most such references will also provide a useful overview of many of the most popular numerical techniques that are currently being used to discretize (convert to algebraic form) the Einstein equations, as well as the main algorithms that are used to solve the resulting discrete equations. These subjects *are* not described below, not least since discussion of the available discretization techniques only makes sense in the context of specific systems PDEs with specific boundary conditions, while there is only space here to describe the *general* mathematical setting for 3 + 1 numerical relativity.

III. THE 3 + 1 SPACETIME SPLIT

At least at the current time, computations in numerical relativity are restricted to the case of *globally hyperbolic* spacetimes. A spacetime (4-dimensional pseudo-Riemannian manifold), $\mathcal{M}_\Sigma$, endowed with a metric, $g_{\mu\nu}$, is globally hyperbolic if there is at least one edgeless, spacelike hypersurface, $\Sigma(0)$, that serves as a Cauchy surface. That is, provided that the initial data for the gravitational field is set consistently on $\Sigma(0)$—so that the four constraint equations are satisfied (see below)—the *entire metric* $g_{\mu\nu}(t, x^i)$ can be determined from the field equations (1) (with appropriate boundary conditions), and thus so can the complete geometric structure of the spacetime manifold.

To be sure, global hyperbolicity *is* restrictive. It excludes, for example, the highly interesting Gödel universe. However, particularly from the point of view of studying asymptotically flat solutions (or solutions asymptotic to any of the currently popular cosmologies), as is usually the case in astrophysics, the requirement of global hyperbolicity is natural.

The 3 + 1 split is based on the complete foliation of $\mathcal{M}_\Sigma$ based on level surfaces of a scalar function, t—the time function. That is, the $t =$ const. slices, are three-dimensional spacelike (Riemannian) hypersurfaces, and, as t ranges from $-\infty$ to $+\infty$, completely fill the spacetime manifold, $\mathcal{M}_\Sigma$. In order for the $\Sigma(t)$ to be everywhere spacelike, t must be everywhere timelike;

$$g_{\mu\nu}\nabla^\mu t\nabla^\nu t < 0 \,. \tag{5}$$

Here ∇_μ is the spacetime covariant derivative operator compatible with the four metric, $g_{\mu\nu}$, thus satisfying $\nabla_\alpha g_{\mu\nu} = 0$, and $g^{\mu\nu}$ is the inverse metric tensor, which satisfies $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$. The reader is reminded that $\delta^\mu{}_\nu$ is a Kronecker delta symbol; that is, $\delta^\mu{}_\nu$ has the value 1 if $\mu = \nu$, and the value 0 otherwise.

Furthermore, the scalar function t is now adopted as *the* temporal coordinate, so that $x^\mu = (t, x^i)$, where the x^i are the three spatial coordinates. As noted implicitly above, since the problem under consideration is a pure Cauchy evolution, the range of t should nominally be infinite, both to the future as well as to the past; that is, the solution domain is

$$-\infty < t < \infty \,, \tag{6}$$

$$|X| \equiv \left(\gamma_{ij}x^i x^j\right)^{\frac{1}{2}} < \infty \,. \tag{7}$$

However, this assumes that one has global existence for arbitrarily strong initial data, which is decidedly *not* always the case in general relativity. Indeed "continued" or "catastrophic" gravitational collapse—that is, the process of black hole formation—signaled, in modern language, by the appearance of a *trapped surface*, inexorably leads to a physical singularity, which—the somewhat vague nature of the singularity theorems of Penrose, Hakwing and others notwithstanding—in actual numerical computations invariably turns out to be "catastrophic" in terms of Cauchy evolution.

Such behaviour in time-dependent nonlinear PDEs is quite familiar in the mathematical community at large, where it is frequently known as *finite time blow up* (or *finite time singularity*). However, despite the fact that such behaviour is one of the most fascinating aspects of solutions of the Einstein equations, the following discussion will be, implicitly at least, restricted to the case of *weak* initial data, that is to initial data for which there *is* global existence.

With the manifold $\mathcal{M}_\Sigma$ sliced into an infinite stack of spacelike hypersurfaces, $\Sigma(t)$, attention shifts to any *single* surface, as well as to the manner in which such a generic surface is embedded in the spacetime.

First, each spacelike hypersurface, $\Sigma(t)$, is itself a 3-dimensional *Riemannian* differential manifold with a metric $\gamma_{ij}(t, x^k)$. (Note that in this discussion, the symbol t is to be understood to represent any specific value of coordinate time.) From this metric, one can construct an inverse metric, $\gamma^{ij}(t, x^k)$, defined, as usual, so that

$$\gamma^{ik}\gamma_{kj} = \delta^i{}_j \,. \tag{8}$$

Associated with the spatial metric, γ_{ij}, is a natural spatial covariant derivative operator, D_i that is compatible with γ_{ij}:

$$D_k\gamma_{ij} = 0 \,. \tag{9}$$

With the spatial metric, γ_{ij}, and its inverse, γ^{ij}, in hand, the standard formulae of tensor analysis can be applied to compute the usual suite of geometrical tensors. All tensors thus computed, and indeed, all tensors defined *intrinsically* to the hypersurfaces $\Sigma(t)$ are called *spatial* tensors, and have their indices (if any) raised and lowered with γ^{ij} and γ_{ij}, respectively.

Thus, the Christoffel symbols of the second kind, $\Gamma^i{}_{jk}$, are given by

$$\Gamma^i{}_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_k\gamma_{lj} + \partial_j\gamma_{lk} - \partial_l\gamma_{jk}\right) \,. \tag{10}$$

Note that these quantities are symmetric in their *last* two indices

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj} \,, \tag{11}$$

and that they can be used, as usual, in explicit calculation of the action of the spatial covariant derivative operator on an arbitrary tensor. In particular, for the special cases of a spatial vector, V^i, and a co-vector (one-form), W_i, one has

$$D_i V^j = \partial_i V^j + \Gamma^j{}_{ik}V^k \,, \tag{12}$$

and

$$D_i W_j = \partial_i W_j - \Gamma^k{}_{ij}W_k \,, \tag{13}$$

respectively.

Given the Christoffel symbols, the components of the spatial Riemmann tensor, denoted here $\mathcal{R}_{ijk}{}^l$, are computed using

$$\mathcal{R}_{ijk}{}^l = \partial_j\Gamma^l{}_{ik} - \partial_i\Gamma^l_{jk} + \Gamma^m{}_{ik}\Gamma^l{}_{mj} - \Gamma^m{}_{jk}\Gamma^l{}_{mi} \,. \tag{14}$$

Finally, the Ricci tensor, $\mathcal{R}^i{}_j$, and Ricci scalar, $\mathcal{R}$, are defined in the usual fashion

$$\mathcal{R}^i{}_j = \gamma^{ik}\mathcal{R}_{kj} = \gamma^{ik}\mathcal{R}_{klj}{}^l \,, \tag{15}$$

$$\mathcal{R} = \gamma^{ij}\mathcal{R}_{ij} \,. \tag{16}$$

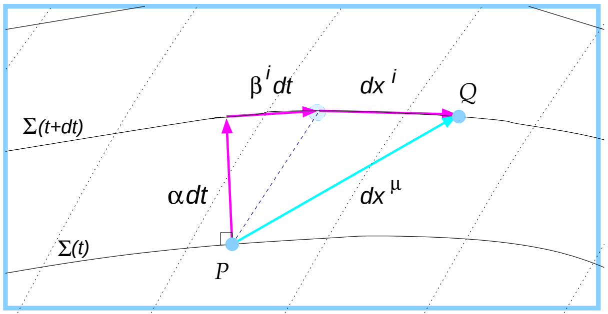

FIG. 1: Spacetime displacement in the $3+1$ approach, following Misner, Thorne and Wheeler (1973). Solid lines represent surfaces of constant time, t; that is, each solid line represents a single spacelike hypersurface, $\Sigma(t)$. Dotted lines denote trajectories of constant spatial coordinate, that is, trajectories with $x^k =$ const. The *lapse function*, $\alpha(t, x^k)$, encodes the (local) ratio between elapsed coordinate time, dt, and elapsed proper time, $d\tau = \alpha dt$, for an observer moving normal to the slices (i.e. for an observer with a four velocity, u^μ, identical to the hypersurface normal, n^μ). Similarly, the shift vector, $\beta^i(t, x^k)$, describes the shift, $\beta^i(t, x^i)dt$, in trajectories of constant spatial coordinate—the dotted lines in the figure—relative to motion perpendicular to the slices. The $3+1$ form of the line element (18) then follows immediately from an application of the spacetime version of the Pythagorean theorem.

The reader should again note that *all* of the tensors just defined "live" on each and every single spacelike hypersurface, $\Sigma(t)$, and are thus known as hypersurface-intrinsic quantities. In particular, the spatial Riemann tensor, $\mathcal{R}_{ijk}{}^l$, which encodes *all* intrinsic geometric information about $\Sigma(t)$, *in no way depends on how the slice is embedded in the spacetime* $\mathcal{M}_\Sigma$.

The next step in the $3+1$ approach involves rewriting the fundamental spacetime line element for the squared proper distance, ds^2, between two spacetime events, $\mathcal{P}$ and $\mathcal{Q}$, having coordinates x^μ and $x^\mu + dx^\mu$ respectively,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \, . \tag{17}$$

As Fig. 1 illustrates, a quick route to the 3+1 decomposition of the above expression, and thus of the tensor $g_{\mu\nu}$ itself, is based on an application of the "4-dimensional Pythagorean theorem". In setting up the calculation, one naturally identifies four functions, the scalar *lapse*, $\alpha(t, x^k)$, and the vector *shift*, $\beta^i(t, x^k)$, that encode the full coordinate (gauge) freedom of the theory. That is complete specification of the lapse and shift is equivalent to completely fixing the spacetime coordinate system.

In light of the above discussion, and again referring to Fig. 1, one readily deduces the 3+1 decomposition of the spacetime line element:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt\right) \left(dx^j + \beta^j dt\right) \, . \tag{18}$$

A rearranged form of this last expression is also often seen in the literature:

$$ds^2 = \left(-\alpha^2 + \beta_k \beta^k\right) dt^2 + 2\beta_k dx^k dt + \gamma_{ij} dx^i dx^j \, . \tag{19}$$

The following useful identifications of the "time-time", "time-space", and "space-space" pieces of the spacetime metric, $g_{\mu\nu}$, follow immediately from (19):

$$g_{00} = -\alpha^2 + \beta^i \beta_i \tag{20}$$
$$g_{0i} = g_{i0} = \beta_i = \gamma_{ik} \beta^k \tag{21}$$
$$g_{ij} = \gamma_{ij} \tag{22}$$

This last relation is an example of a useful general result; the purely spatial components, $Q_{ijk\cdots}$, of a completely covariant, but otherwise arbitrary, *spacetime* tensor, $Q_{\alpha\beta\gamma\cdots}$, constitute the components of a completely covariant *spatial* tensor.

A straightforward calculation, which provides a good exercise in the use of the 3+1 calculus, yields the following equally useful identifications for various pieces of the *inverse* spacetime metric: $g^{\alpha\beta}$

$$g^{00} = -\alpha^{-2} \tag{23}$$
$$g^{0i} = g^{i0} = \alpha^{-2} \beta^i \tag{24}$$
$$g^{ij} = \gamma^{ij} - \alpha^{-2} \beta^i \beta^j \tag{25}$$

Since the Einstein field equations *are* equations with, loosely speaking, geometry on one side and matter on the other, tensors built from matter fields must also be decomposed. In particular, it is conventional to define tensors, ρ, j_i and S_{ij} that result from various projections of the spacetime stress energy tensor, $T_{\mu\nu}$, onto the hypersurface:

$$\rho \equiv n_\mu n_\nu T^{\mu\nu} \, , \tag{26}$$
$$j_i \equiv -n_\mu T^\mu{}_i \, , \tag{27}$$
$$S_{ij} \equiv T_{ij} \, . \tag{28}$$

For observers with four velocities u^μ equal to n^μ, and *only* for those observers with $u^\mu = n^\mu$, the above quantities have the interpretation of the locally and instantaneously measured energy density, momentum density and spatial stresses respectively. As with the geometric quantities, all of the matter variables, ρ, j_i, and S_{ij} defined in (26–28) are spatial tensors and thus have their indices (if any) raised and lowered with the 3-metric. Note that the identification $S_{ij} = T_{ij}$ is another illustration of the general result mentioned in the context of the previous identification of γ_{ij} and g_{ij}.

Finally, observing that time parameters are naturally defined in terms of level surfaces (equipotential surfaces), it should be no surprise that the covariant components, n_μ, of the hypersurface normal field,

$$n_\mu = (-\alpha, 0, 0, 0) \, , \tag{29}$$

are simpler than the components, n^μ of the normal itself,

$$n^\mu = \left(\alpha^{-1}, \alpha^{-1}\beta^i\right) , \quad (30)$$

and, in fact, equation (29) can also be deduced from a quick study of Fig. 1.

In the 3+1 approach, in addition to the 3-metric, $\gamma_{ij}(t, x^k)$, and coordinate functions, $\alpha(t, x^i)$ and $\beta(t, x^i)$, it is convenient to introduce an additional rank-2 symmetric spatial tensor, $K_{ij}(t, x^k)$, known as the extrinsic curvature (or second fundamental form). This additional tensor is analogous to a time derivative of $\gamma_{ij}(t, x^k)$, or, from a Hamiltonian perspective, to a variable that is dynamically conjugate to $\gamma_{ij}(t, x^k)$.

As the name suggests, the extrinsic curvature describes the manner in which the slice $\Sigma(t)$ is embedded in the manifold (to be contrasted with $\mathcal{R}_{ijk}{}^l$ defined by (14) which is, as mentioned previously, completely insensitive to the manner in which the hypersurface is embedded in $\mathcal{M}_\Sigma$).

Geometrically, K_{ij} is computed by calculating the spacetime gradient of the normal covector field, n_μ, and projecting the result on to the hypersurface,

$$K_{ij} = -\frac{1}{2}\nabla_i n_j , \quad (31)$$

where it must be stressed that ∇_μ is the *spacetime* covariant derivative operator compatible with the 4-metric, $g_{\alpha\beta}$; that is, $\nabla_\mu g_{\alpha\beta} = 0$. A straightforward tensor calculus calculation then yields the following, which can be viewed as a *definition* of the K_{ij}:

$$K_{ij} = \frac{1}{2\alpha}\left(\partial_t \gamma_{ij} + D_i\beta_j + D_j\beta_i\right) . \quad (32)$$

Here, D_i is the spatial covariant metric, compatible with γ_{ij} ($D_k\gamma_{ij} = 0$), that was defined previously. Observe that this equation can be easily solved for $\partial_t\gamma_{ij}$ (this will be done below), and thus, in the 3+1 approach it is (32) that is the origin of the evolution equations for the 3-metric components, γ_{ij}.

IV. EINSTEIN'S EQUATIONS IN 3 + 1 FORM

A. The Constraint Equations

As is well known, as a result of the coordinate (gauge) invariance of the theory, general relativity is overdetermined in a sense completely analogous to the situation in electrodynamics with the Maxwell equations. One of the ways that this situation is manifested is via the existence of the constraint equations of general relativity. Briefly, starting from the naive view that the ten metric functions, $g_{\mu\nu}(t, x^i)$, that completely determine the spacetime geometry are all *dynamical*—that is that they satisfy second-order-in-time equations of motion—one finds that the Einstein equations do *not* provide dynamical equations of motion for the lapse, α, or the shift, β^i. Rather, four of the field equations (1) are equations of *constraint* for the "true" dynamical variables of the theory, $\{\gamma_{ij}, \partial_t\gamma_{ij}\}$, or, equivalently, $\{\gamma_{ij}, K^i{}_j\}$. Note that in the following, the mixed form, $K^i{}_j$, is at times used—again by convention—as the principal representation of the extrinsic curvature tensor (instead of K_{ij} as previously, or K^{ij}).

Thus, four of the components of (1) can be written in the form

$$C^\mu\left(\gamma_{ij}, K^i{}_j, \partial_k\gamma_{ij}, \partial_l\partial_k\gamma_{ij}, \partial_k K^i{}_j\right) = T^\mu , \quad (33)$$

where T^μ depends only on the matter content in the spacetime. Note that in addition to having *no* dependence on $\partial_t^2\gamma_{ij}$, the constraints are also independent of α and β^i.

If the Einstein equations (1) are to hold throughout the spacetime, then the constraints (33) must hold on each and every spacelike hypersurface, $\Sigma(t)$, including, crucially, the initial hypersurface, $\Sigma(0)$. From the point of view of Cauchy evolution, this means that the 12 functions, $\{\gamma_{ij}(0, x^k), K^i{}_j(0, x^k)\}$, constituting the gravitational part of the initial data, are *not* completely freely specifiable, but must satisfy the 4 constraints

$$C^\mu\left(\gamma_{ij}(0, x^k), K^i{}_j(0, x^k), \cdots\right) = T^\mu(0, x^k) . \quad (34)$$

However, provided initial data that *does* satisfy the equations is chosen, then—as consistency of the theory demands—the dynamical equations of motion for the $\{\gamma_{ij}, K^i{}_j\}$ (equations (37-38) below) guarantee that the constraints will be satisfied on all future (or past) hypersurfaces, $\Sigma(t)$. In this internal self-consistency, the geometrical Bianchi identities, $\nabla_\mu G^{\mu\nu} = 0$, and the local conservation of stress energy, $\nabla_\mu T^{\mu\nu} = 0$, play crucial roles.

In the 3+1 approach, as one would expect, the constraint equations further naturally subdivide into a scalar equation

$$\mathcal{R} - K_{ij}K^{ij} + K^2 = 16\pi\rho , \quad (35)$$

and a (spatial) vector equation

$$D_j K^{ij} - D^i K = 8\pi j^i , \quad (36)$$

where the energy and momentum densities, ρ and $j^i = \gamma^{ik} j_k$, are given by (26–28). Equations (35) and (36) are often known as the Hamiltonian and momentum constraint, respectively, not least since the behaviour of their solutions as $X \equiv \sqrt{\gamma_{ij}x^i x^j} \to \infty$ encodes the conserved mass and linear momentum (4 numbers) that can be defined in asymptotically flat spacetimes.

In a general $3+1$ coordinate system, and with an appropriate choice of variables, the constraints can be written as a set of quasi-linear elliptic equations for four of the $\{\gamma_{ij}, K^i{}_j\}$ (or, more properly, for certain algebraic combinations of the $\{\gamma_{ij}, K^i{}_j\}$). Thus, especially for 2D and

3D calculations, the setting of initial data for the Cauchy problem in general relativity is itself a highly non-trivial mathematical and computational exercise. Those readers wishing more detail on this subject are directed to the comprehensive review by Cook (2000).

B. The Evolution Equations

As discussed above, in the 3 + 1 form of the Einstein equations (1), the spatial metric, γ_{ij}, and the extrinsic curvature, $K^i{}_j$, are viewed as the dynamical variables for the gravitational field. The remainder of the 3 + 1 equations are thus two sets of 6 first-order-in-time evolution equations; one set for γ_{ij},

$$\partial_t \gamma_{ij} = -2\alpha\gamma_{ik}K^k{}_j + \beta^k\partial_k\gamma_{ij} + \gamma_{ik}\partial_j\beta^k + \gamma_{kj}\partial_i\beta^k \,, \quad (37)$$

and the other set for $K^i{}_j$,

$$\partial_t K^i{}_j \;=\; \beta^k\partial_k K^i{}_j - \partial_k\beta^i K^k{}_j + \partial_j\beta^k K^i{}_k - D^iD_j\alpha + \alpha\left(\mathcal{R}^i{}_j + KK^i{}_j + 8\pi\left(\frac{1}{2}\delta^i{}_j\left(S-\rho\right) - S^i{}_j\right)\right) . \quad (38)$$

As also noted previously, the evolution equations (37) for the spatial metric components, γ_{ij}, follow from the definition of the extrinsic curvature (31). The derivation of the equations for the extrinsic curvature, on the other hand, require lengthy, but well documented, manipulations of the spatial components of the field equations (1).

C. The (Naive) Cauchy Problem

A naive statement of the Cauchy problem for 3 + 1 numerical relativity is thus as follows: Fix a specified number, N, of matter fields $\xi^A(t, x^k), A = 1, 2, \cdots N$, all minimally coupled to the gravitational field, with a total stress tensor, $T_{\mu\nu}$, given by

$$T_{\mu\nu} = \sum_{A=1}^{N} T^A_{\mu\nu} \,, \quad (39)$$

where $T^A_{\mu\nu}$ is the stress tensor corresponding to the matter field ξ^A. Choose a topology for $\Sigma(0)$ (for example, $\mathcal{R}^3$ with asymptotically flat boundary conditions; T^3, with *no* boundaries etc.) This also fixes the topology of $\mathcal{M}_\Sigma$ to be $\mathbb{R}\times$ the topology of $\Sigma(0)$.

Next, freely specify 8 of the 12 $\{\gamma_{ij}(0, x^k), K^i{}_j(0, x^k)\}$, as well as initial values, $\xi^A(0, x^k)$, for the matter fields. Then determine the remaining 4 dynamical gravitational fields from the constraints (35) and (36). This completes the initial data specification.

One must now choose a prescription for the kinematical (coordinate) functions, α and β^i, so that either explicitly or implicitly, they are completely fixed; for the case of implicit specification this may well mean that the coordinate functions themselves will satisfy PDEs, which, furthermore, can be of essentially *any* type in practice (i.e. elliptic, hyperbolic, parabolic, ...). Finally, with consistent initial data, $\{\gamma_{ij}(0, x^k), K^i{}_j(0, x^k); \xi_A(0, x^k)\}$, in hand, and with a prescription for the coordinate functions, the evolution equations (37) and (38) can be used to advance the dynamical variables forward or backward in time.

The above description is naive since, apart from a consistent mathematical specification, the most crucial issue in the solution of a time-dependent PDE as a Cauchy problem, is that the problem be *well posed*. Roughly speaking, this means that solutions do not grow without bound ("blow up") without physical cause, and that small, smooth changes to initial data yield correspondingly small, smooth changes to the evolved data. In short, the Cauchy problem must be *stable*, and whether or not a particular subset of the equations displayed in this section yields a well posed problem is a complicated and delicate issue, especially in the generic 3D case. The reader is thus again cautioned against blind application of *any* of the equations displayed in this article.

D. Boundary Conditions

In principle, because all spacelike hypersurfaces, $\Sigma(t)$, in a pure Cauchy evolution are edgeless—and provided that the initial data $\{\gamma_{ij}(0, x^k), K^i{}_j(0, x^k); \xi_A(0, x^k)\}$ is consistent with asymptotic flatness, or whatever other condition is appropriate given the topology of the $\Sigma(t)$—there are essentially *no* boundary conditions to be imposed on the dynamical variables, $\{\gamma_{ij}(t, x^k), K^i{}_j(t, x^k)\}$ during Cauchy evolution. Note that asymptotic flatness generally requires that

$$\lim_{X\to\infty} \gamma_{ij} = f_{ij} + O\left(\frac{1}{X}\right) \,, \quad (40)$$

and

$$\lim_{X\to\infty} K^i{}_j = O\left(\frac{1}{X^2}\right) \,, \quad (41)$$

where X is defined by

$$X \equiv \sqrt{\gamma_{ij}x^ix^j} \quad (42)$$

as previously, and f_{ij} is the flat 3-metric. Similarly, should the lapse, α, and shift, β, be constrained by elliptic partial differential equations—as is frequently the case in practice—then the only natural place to set boundary conditions is at spatial infinity, and then, provided that the frame at spatial infinity is inertial, with coordinate time t measuring proper time, one should have

$$\lim_{X\to\infty} \alpha = 1 + O\left(\frac{1}{X}\right) \,, \quad (43)$$

and

$$\lim_{X\to\infty} \beta^i = O\left(\frac{1}{X}\right) . \qquad (44)$$

It is critical to note at this point, however, that in the vast bulk of past and current work in numerical relativity, including most of the ongoing work in 3D, the Einstein equations (1) have been solved, not as a pure Cauchy problem, but as a mixed initial-value/boundary-value (IBVP) problem. That is, in the discretization process in which the continuum equations (1) are replaced with algebraic equations, the continuum domain (6-7) is typically replaced with a truncated spatial domain

$$|x^i| \leq X^i_{\max} \qquad (45)$$

where the $X^i_{\max}$ are *a priori* specified constants (parameters of the computational solution) that define the extremities of the "computational box". As one might expect, the theory underlying stability and well-posedness of IBVP problems—especially for differential systems as complicated as (1) is even more involved than for the pure initial-value case, and is another very active area of research in both mathematical *and* numerical relativity (see, for example, Friedrich and Nagy (1999)).

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