AN OVERVIEW AND REPRODUCTION OF BARTNIK AND MCKINNON'S DISCOVERY OF PARTICLELIKE SOLUTIONS TO THE EINSTEIN-YANG-MILLS EQUATIONS

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1. Introduction

In their 1988 paper[1] "Particlelike Solutions of the Einstein-Yang-Mills Equations", Bartnik and McKinnon showed that an SU(2) Yang-Mills field coupled to gravity would allow non-singular, static solutions- hence, the existence of particlelike solutions. These solutions were parametrized by the number of zero-crossings in the Yang-Mills potential. This was a novel discovery because previously it had been shown that neither the vacuum Einstein equations nor the pure Yang-Mills equations, allowed nontrivial, static, globally regular solutions [5, 6]. Thus it was conjectured that perhaps the coupled Einstein-Yang-Mills (EYM) equations did not permit non-trivial, globally regular, static solutions. However, Bartnik and McKinnon showed that the opposite is true.

The crux of Bartnick and McKinnon's solutions lies in finding globally regular solutions to a system of time-independent ODEs via a shooting method. Though in their seminal paper they chose to begin integration "near zero"[1] , a later paper[3] by Breitlohner, Forgaçs, and Maison showed that the equations could be desingularized via a change of variables, thus allowing one to begin integration at r=0 and hence obtain a truly globally regular solution.

In the following work I will present my own verification of the existence of globally regular, static solutions to the Einstein-Yang-Mills equations, following the route described in [3]. In Section 2 I introduce the Einstein-Yang-Mills equations and derive the equations that Bartnik and McKinnon used in their paper. Section 3 works through the change of coordinates used to desingularize the equations at r=0. The method for numerically integrating the equations is described in Section 4. Finally, the particlelike solutions are found, Section 5, and their phenomenology discussed, Section 6. Section 7 concludes.

2. The Einstein-Yang-Mills Equations

2.1. **Background.** Why look for solutions to a spherically symmetric SU(2) Yang-Mills field coupled to gravity? An interesting counter-question, to which we now know the answer, is: do the EYM questions allow nontrivial globally regular, static solutions? Since the EYM equations couple the Yang-Mills equations with Einstein's equations, we might be inclined to make an educated guess based on what each of the constituent theories predict. In 1976 it was shown by Deser [6] that the pure Yang-Mills equations do not admit nontrivial, static, globally regular solutions. Similarly, by Lichnerowicz's theorem[1], the only static solution to Einstein's equations that is globally regular is vacuum- the trivial solution. So if both theories predict that no nontrivial, globally regular, static solutions exist, one might expect the resulting coupled theory to behave similarly. Hence, the suprise when Bartnik and McKinnon announced the discovery of their solutions. As they state in their paper

> "The gravitational interaction cannot be dismissed as too weak to be of consequence- the existence of these solutions depends essentially on the interaction between the YM and Einstein equations."

It appears that by coupling the two theories the existence of particlelike solutions is introduced. This existence relies crucially on the balance between the gravitational and Yang-Mills forces. Let us begin to analyze the foundation of this delicate balance.

[1]They began integration at r=0.01 because of an r=0 singularity in the equations

2.2. **The Form of the Metric.** For the following work, we will take as an ansatz due to Bergman that the form of the line element is

$$ds^2 = A^2\mu dt^2 - \frac{dr^2}{\mu} - r^2 d\Omega^2 \tag{1}$$

where $A = A(r)$ is undetermined, and $\mu = \mu(r)$ can be related to the "mass function" $m(r)$ by

$$\mu(r) = 1 - 2m(r)/r$$

Eq. (1) can be derived from the metric of an arbitrary, static, and spherically symmetric spacetime [7]:

$$\begin{aligned} ds^2 &= -T^{-2}dt^2 + R^2 dr^2 - r^2(R)d\Omega^2 \\ &= e^{2\nu(R)}dt^2 + e^{2\lambda(R)}dR^2 - r^2(R)d\Omega^2 \end{aligned} \tag{2}$$

The second form is Bergman's convention [2].

2.3. **The SU(2) Gauge Choice and the Lagrangian.** Witten provided a seminal ansatz for the SU(2) spherically symmetric gauge connection A in his 1977 paper [8], and we use this ansatz as our starting point. Witten found that the gauge connection for a spherically symmetric SU(2) Yang-Mills field can be written as:

$$A = a\tau_3 dt + b\tau_3 dr + (c\tau_1 + d\tau_2)d\theta + (cot\theta\tau_3 + c\tau_2 - d\tau_1)sin\theta d\phi$$

where $\tau_{1,2,3}$ correspond to the spherical polar unit vectors $\tau_{\theta,\phi,r}$ used as a basis for SU(2)[2]. If we rename the variables in the following manner: $a = A_0$, $b = A_1$, $c = \phi_1$, $d = \phi_2$, we can write the SU(2) connection for the Yang-Mills field $W_\mu^a(\tau_1, \tau_2, \tau_3)$ as:

$$\begin{aligned} W_t^a &= (0, 0, A_0), \qquad W_\theta^a = (\phi_1, \phi_2, 0) \\ W_r^a &= (0, 0, A_1), \qquad W_\psi^a = (-\phi_2 sin\theta, \phi_1 sin\theta, cos\theta) \end{aligned} \tag{3}$$

Bartnik and McKinnon describe the Lagrangian using forms as:

$$\mathcal{L} = (-\mathcal{R} + |\mathcal{F}|^2)\sqrt{g} \tag{4}$$

where

$$F = dA + A \wedge A$$

and R is a metric coefficient.

This results in a Lagrangian of:

$$\begin{aligned} \mathcal{L} =& A(r)\Bigg[\frac{1}{2G}(\mu + r\mu' - 1) + \\ & \frac{1}{g^2}\Bigg(-\frac{r^2}{2A^2}{A'}_0^2 + \mu[(\phi_1' + A_1\phi_2)^2 + (\phi_2' - A_1\phi_1)^2] \\ & - \frac{A_0^2}{\mu A^2}(\phi_1^2 + \phi_2^2) + \frac{(1 - \phi_1^2 - \phi_2^2)^2}{2r^2}\Bigg)\Bigg] \end{aligned} \tag{5}$$

where g and G are gauge and gravitational coupling constants respectively, and ϕ_1 is now written as W. Since the field W_μ^a is now a function of W only, from here on we refer to W as the field.

Various simplifications due to the U(1) gauge freedom of Witten's ansatz, Eq. (3), and to the field equations themselves allow us to set $A_1 = 0$ and $\phi_2 = 0$. Finally A_0 is set to zero because it is known that no regular solutions exist with $A_0 \neq 0$, except for Reissner-Nordstrom solutions. This results in a tremendous simplification of the Lagrangian, now in its final form for the remainder of this paper:

$$\mathcal{L} = A(r)\left[\frac{1}{2G}(\mu + r\mu' - 1) + \frac{1}{g^2}\left(\mu {W'}^2 + V\right)\right] \tag{6}$$

where

$$V = \frac{(1 - W^2)^2}{2r^2}$$

[2]The spherical polar unit vectors are the spherical projection of the cartesian Pauli matrices; the usual basis for SU(2)

Finally, for completeness, we state the definition of the action:

$$S = -\int dr\mathcal{L} \tag{7}$$

2.4. **Equations of Motion.** If we vary the action, Eq. (7), with respect to a variable x we find that the variable satisfies the Euler-Lagrange equations of motion given by:

$$\frac{d}{dr}\left(\frac{\partial L}{\partial x'}\right) - \frac{\partial L}{\partial x} = 0$$

Upon the following rescaling,

$$S \rightarrow g\sqrt{G}S$$
$$r \rightarrow \frac{r\sqrt{G}}{g}$$

and varying Eq. (7) with respect to μ, the field W, and A respectively, yields the equations of motion[3]:

$$\boxed{A^{-1}A' = \frac{2W'^2}{r}} \tag{8}$$

$$\boxed{(\mu W')' = \frac{W(W^2-1)}{r^2} - \frac{2\mu W'^3}{r}} \tag{9}$$

$$\boxed{\mu' = \frac{1}{r}\Big(1 - \mu - 2(\mu W'^2 + V)\Big)} \tag{10}$$

It is now up to us to integrate these manifestly time-independent equations for all r and determine if any non-trivial configurations of the field W exist that are globally regular. First it is useful to regularize the equations so that we have no singularities caused by the $\frac{1}{r}$ terms evident in Eqns. (8,10). Then our final set of equations can be numerically integrated on a computer. The de-singularized equations are derived via a change of variables in the next section.

3. Regularization

De-singularizing Eqns. (8-10) will proceed via two sets of variable transformations. The first is relatively simple and clean, and was done by hand. The second is tedious and messy, and was done in Maple[4]; however the final simplified results are shown.

3.1. **The First Transformation.** The first transformation proceeds in the following manner. Introduce the following variables:

$$w_1 = \frac{1-W}{r^2}, \qquad w_2 = \frac{\mu W'}{r}, \qquad w_3 = \frac{1-\mu}{r^2} \tag{11}$$

or, alternatively,

$$W = 1 - w_1 r^2, \qquad W' = \frac{r w_2}{\mu}, \qquad \mu = 1 - w_3 r^2 \tag{12}$$

Then, upon differentiating the first of Eqns. (11) we have:

$$\boxed{r w_1' = \frac{-w_2}{1 - w_3 r^2} - 2w_1} \tag{13}$$

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[3]See Appendix A for detailed calculations.
[4]See Appendix B for details.

Upon substituting Eqns. (12) into Eqns. (9,10), respectively, we find two more equations:

$$\boxed{rw_2' = -2w_1 - w_2 + r^2\left[3w_1^2 - r^2w_1^3 - \frac{2w_2^3}{\mu^2}\right]} \tag{14}$$

and

$$\boxed{rw_3' = -3w_3 + \frac{2w_2^2}{1 - w_3r^2} + 4w_1^2 - 4r^2w_1^3 + r^4w_1^4} \tag{15}$$

3.2. **The Second Transformation.** Now if we make the second change of variables according to:

$$w_1 = u_1 + v_1, \qquad w_2 = -2u_1 + v_1, \qquad w_3 = 2v_2 + 2(2u_1^2 - v_1^2) \tag{16}$$

we find a suitably desingularized version of the equations in the following form:

$$ru_1' = r^2 f_1 \tag{17}$$

$$rv_1' = -3v_1 + r^2 g_1 \tag{18}$$

$$rv_2' = -3v_2 + r^2 g_2 \tag{19}$$

Where[5] f_1, g_1, and g_2 are very ugly polynomials in u, v, r^2, and $\frac{1}{\mu}$.

According to Proposition 1 in [3] then, a solution to Eqns. (21-23) exists of the form

$$u_1 = b + O(r^2), \qquad v_1 = O(r^2), \qquad v_2 = O(r^2) \tag{20}$$

Note that Eq. (20) clearly defines a family of solutions based on one parameter, b- the initial value of u_1. So by integrating Eqns. (21-23), with the initial conditions

$$u_1(0) = b, \qquad v_1(0) = 0, \qquad v_2(0) = 0,$$

and reconstructing the field, W, from our new variables via

$$W = 1 - w_1r^2 = 1 - (u_1 + v_1)r^2,$$

we should be able to search for solutions, by varying b, that are nontrivial and globally regular[6].

4. Integration

4.1. **Integration Implementation.** The integration algorithm implemented to solve Eqns. (21-23) was a tolerance-checking variable stepsize Runge-Kutta integrator written in Fortran. The equations solved were not actually Eqns. (21-23), but Eqns. (21-23) divided by r:

$$u_1' = rf_1 \tag{21}$$

$$v_1' = \frac{-3v_1}{r} + rg_1 \tag{22}$$

$$v_2' = \frac{-3v_2}{r} + rg_2 \tag{23}$$

The complete system of equations, with f_1, g_1, and g_2 explicitly solved for, was derived in Maple, exported to Fortran from Maple via its Fortran package, and then directly copied into the Runge-Kutta integrator program, **bm**- see Appendix B for a link to the details of the calculation. At $r = 0$, v_1' and v_2' were set to zero, as should be expected since they are both $O(r^2)$.

The most important arguments to the program are rmax and u1_0. rmax controls the length of integration, and u1_0 is the parameter b. A separate program can call **bm** and run a bisection search based on how the field behaves for a particular value of b. Since the bisection search is fundamental to finding solutions, let's take a look at it more closely.

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[5] See Appendix B for Maple output.

[6] Though Eqns. (21-23) are difficult for humans to interpret, the callous transistors in a computer are completely ambivalent to their form. Therefor, by taking the messy Maple output and exporting it directly to computer code, we can create a program to numerically integrate Eqns. (21-23). The tedious equation juggling, and number crunching aside, we are left with output from the integration that we can plot and put into more aesthetically pleasing forms.

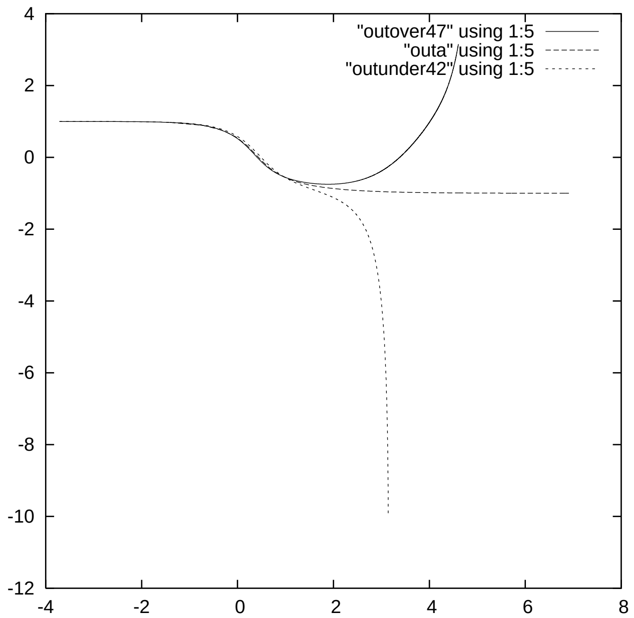

FIGURE 1. W vs. ln r for three numerically integrated solutions corresponding to b = 0.42, 0.45371627277, and 0.47.

4.2. **Bisection Search.** If we are searching for solutions that do not blow up at infinity, we need to tune the parameter b accordingly. Perhaps it could be that all solutions blow up at infinity; however, after running **bm** on a variety of initial values, the following behavior is found: b=0.47 blows up positively, while b=0.42 blows up to negatively. Thus, we have a bracket [low,high], or [0.42, 0.47], in which the solution must switch the direction that it blows up. By bisecting this interval, and running the program for b=0.455, we find that the solution remains flat until larger r, where it eventually blows up positively again. Iterative bisections of the [low,high] interval allow us to close in in a solution that is asymptotically flat and manifestly globally regular. The bisection search described above is illustrated in Fig. 1.

One can clearly see the behavior of the field on either end of the bisection search illustrated in Fig. 1. In this manner we search for functions that do not blow up for large r. The asymptotically flat function plotted is very close to the n=1 Bartick McKinnon solution, achieved with a value of b=0.45371627277.

5. PARTICLELIKE SOLUTIONS

Bartnik and McKinnon found that not only one, but an entire set of nontrivial, globally regular solutions exist. Instead of settling for just b=0.45371627277, they found that b=0.65172552552 was also a solution, as was b=0.69704005033. The distinguishing characteristic amongst these solutions is the number of times the value of the field is zero- so Bartnik and McKinnon classified the solutions based on this number. Thus, the n=1 solution plotted in Fig. 1 crosses 0 once. Fig. 2 shows the first three solutions corresponding to b= 0.45371627277, 0.65172552552, and 0.69704005033 respectively.

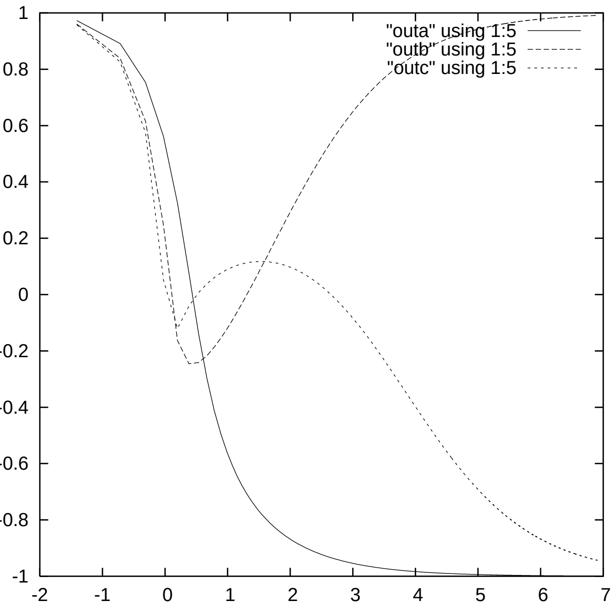

FIGURE 2. The first three globally regular Bartnik McKinnon solutions with n=1,2,3 zero crossings respectively. W vs. ln r

Breitlohner et al. provide numerical values for b corresponding to solutions up to n=10. They also discuss the asymptotic solution which is oscillatory and has b=0.70642037705. This describes an altogether different solution, since it is asymptotically oscillatory, however we will see in the following section that all solutions share similar characteristics.

6. Phenomenology

Fig. 3 shows the n=1, 3, and 5 Bartnik McKinnon solutions. Phenomenologically, they all contain three regions. Region I is an area where the field is nearly equal to 1. Region 2 is a transition region, where the field is between -0.2 and 0.2, and for purposes of discussion is “near zero”. In Region III the field approaches -1 for odd n, or +1 for even n.

Bartnik and McKinnon showed that under the asymptotic conditions $r \to \infty$, the two explicit solutions satisfying $R(r) \to 1$, $T(r) \to 1$, and $m(r) \to M < \infty$ are the Schwarzchild metric and the Reissner-Nordstrom metric.

Paraphrasing Bartnik and McKinnon:

> In the regime where W=1, the metric is Schwarzchild with vanishing Yang-Mills curvature[7]...
> If W=0 the metric describes the Reissner-Nordstrøm metric with mass M, electric charge e, magnetic charge g=1, and non-zero Yang-Mills curvature.

[7]Defined in Section 2.3

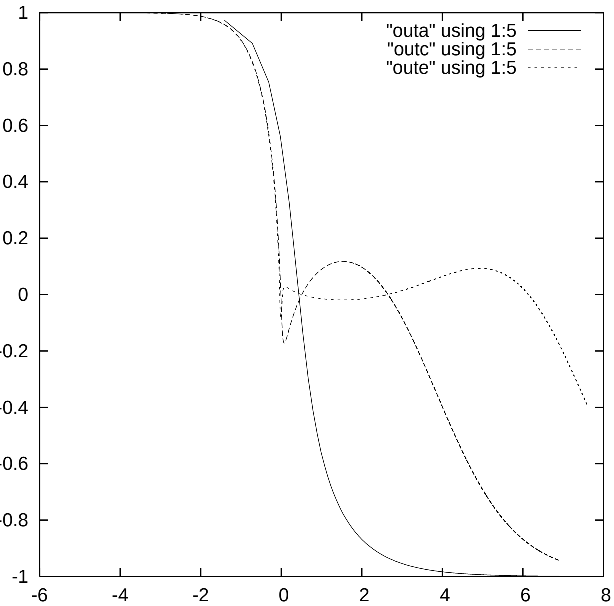

FIGURE 3. The n=1, 3, and 5 Bartnik McKinnon solutions; W vs. ln r

Following Bartnik and McKinnon's advice we can describe the far-field regime, where $W = \pm 1$, as asymptotically Schwarzchild. In Region II, the field oscillates about 0, so we might characterize it as resembling Reissner-Nordstrøm. Bartnik and McKinnon go further as to say it has a Dirac monopole curvature source. The interior region, Region I, is the region of highest energy density and ends abruptly. Interestingly, the radius at which Region I ends decreases slightly with increasing n, whereas the radius where Region II ends increases by an order of magnitude or more with increasing n.

7. Conclusion

The results of Bartnik and McKinnon's paper [1] have been reproduced following the method prescribed by Breitlohner et al. [3]. By way of Witten's spherically symmetric gauge-connection ansatz [8], a static spherically symmetric spacetime ansatz, and a Lagrangian taken as an ansatz, we have ansatz'ed ourselves particlelike solutions. These are the now well-known Bartnik-McKinnon solutions. They are parametrized by the number of times the Yang-Mills field component W becomes zero. The solutions all have three characteristic regions:

(1) A small r region, where $W \cong 1$ and the energy density is highest.
(2) A transition region, where W oscillates (non-periodically) about 0 and the metric is approximately Reissner-Nordstrøm with Dirac monopole curvature source.
(3) A far-field region, where $W = \pm 1$ and the metric is approximately Schwarzchild.

While the stability of the solutions has not been addressed, Bartnik and McKinnon themselves could only hope that at best the n=1 and n=2 solutions may be stable. It has been shown by Zhou [9] that all of the Bartnik-McKinnon solitons are unstable. Furthermore, in [4] Choptuik showed that the Bartnik-McKinnon solutions are only one facet[8] of a larger picture describing the critical behavior of black holes.

Further work that should be done is:

(1) Work through the form-based derivation of the Yang-Mills curvature described in Bartnik and McKinnon's paper [1].
(2) Follow Witten's ansatz for a spherically symmetric SU(2) Yang-Mills field.
(3) Understand how $W = 1$ implies that the metric is Schwarzchild and $W = 0$ that it is Reisner-Nordstrøm.

[8]The Bartnik-McKinnon solutions correspond to Type I critical solutions.

Appendix A. Deriving Equations of Motion

Appendix B. The Second Coordinate Transformation

As of April 26, 2004, the Second Coordinate Transformation is worked out in detail on my webpage at: http://laplace.physics.ubc.ca/People/roland/Papers/bartnik/Eq20.html.

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